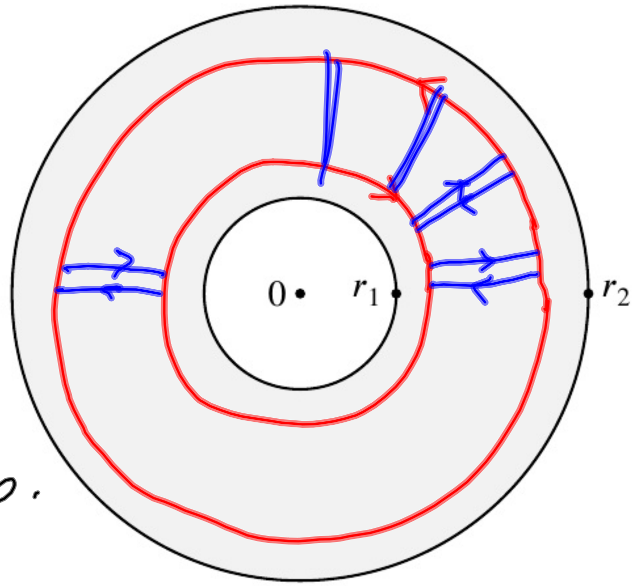
Lemma. *If f is analytic in the annulus $\{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$, then the value of the integral*

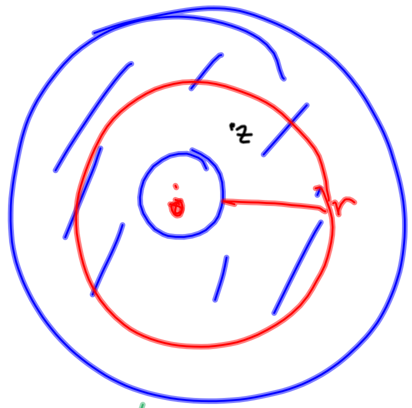
$$\int_{|z|=r} f(z)\,dz \qquad \textit{(counterclockwise orientation)}$$

is not necessarily zero but is independent of the value of r between r_1 and r_2.

Add and subtract integrals over line segments to reduce to a sum of integrals over boundaries of starshaped regions.
Then apply theorem from last time.



Theorem. *If f is analytic in the annulus $\{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$, then there exist a function f_1 analytic when $|z| > r_1$ and a function f_2 analytic when $|z| < r_2$ such that $f = f_2 - f_1$ in the annulus.*



$$\frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}\,dw$$

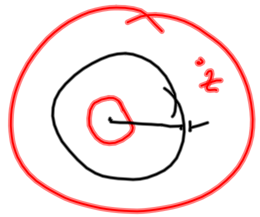
$:= f_2(z)$ where $r_1 < |z| < r < r_2$

or where $|z| < r_2$

and $|z| < r$ and $r_1 < r < r_2$

Similarly define

$$f_1(z) = \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}\,dw$$



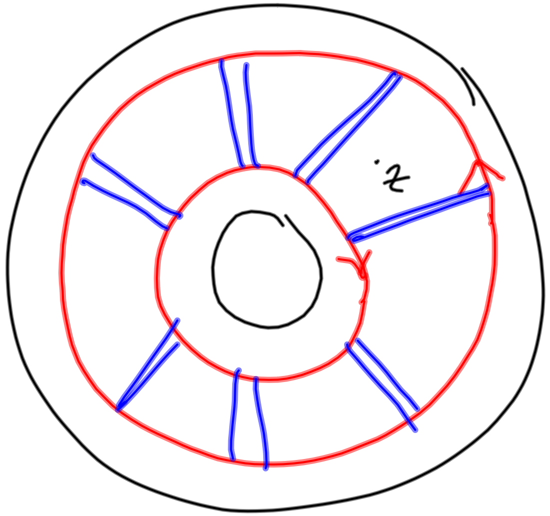
where $r_1 < r < |z| < r_2$

or even $r < |z|$ as long as $r_1 < r < r_2$

Why is $f = f_2 - f_1$ in the annulus?

$f_2(z) - f_1(z) =$

$$\frac{1}{2\pi i} \int_{\text{two circles}} \frac{f(w)}{w-z}\,dw$$



$$= \frac{1}{2\pi i} \int_{\substack{\text{one small starshaped} \\ \text{region containing } z}} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i} \int_{\substack{\text{rectangle} \\ \text{containing } z}} \frac{f(w)}{w-z}\,dw$$ by adding and subtracting some additional line segments

$= f(z)$ by Cauchy integral representation for rectangles.

Corollary. *If f is analytic in the annulus $\{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$, then f can be expanded in a Laurent series*

$$\sum_{n=-\infty}^{\infty} c_n z^n$$

that converges absolutely and uniformly on compact subsets of the annulus.

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}}$$

$$= \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n \quad \text{if } \left|\frac{z}{w}\right| < 1$$

$$f_2(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z}\, dw \qquad |z| < r = |w|$$

$$= \sum_{n=0}^{\infty} z^n \overbrace{\left(\frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w^{n+1}}\, dw \right)}^{c_n}$$

(using Weierstrass M-test to justify uniform convergence and hence justify interchanging sum with the integral)

For f_1, write

$$\frac{1}{w-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{w}{z}}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n \quad \text{when } |w| < |z|$$

etc.

$$c_n = \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w^{n+1}}\,dw$$

works for n positive
and also n negative

Special case: when the annulus is a disk, then $f_1 \equiv 0$ and $f = f_2$
so f has a Taylor series expansion.

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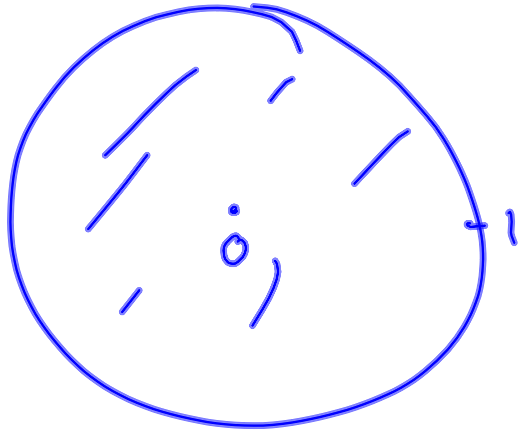
$$f(z) = \frac{1}{z(z-1)(z-2)}$$



three subproblems