Some theorems about the range of *entire* functions

Theorem (Picard's "little theorem"). *The range of a nonconstant entire function is either* $\mathbb{C}$ *or* $\mathbb{C} \setminus \{$*one point*$\}$.

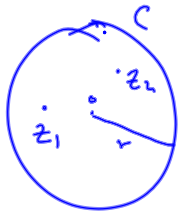
Corollary of Picard's big theorem.

A weaker theorem that we can prove now:

Theorem (Liouville). *The range of a nonconstant entire function is unbounded.*

Contrapositive: A bounded entire function has to be constant.

Proof Fix two points z_1 and z_2.



$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_1} - \frac{f(w)}{w - z_2} \, dw$$

$$= \frac{(z_1 - z_2)}{2\pi i} \int_C \frac{f(w)}{(w - z_1)(w - z_2)} \, dw$$

Absolute value of $\int_C$ is bounded by

$$\frac{\text{constant}}{r^2} \cdot (\underbrace{\text{length of path}}_{2\pi r}) \longrightarrow 0$$

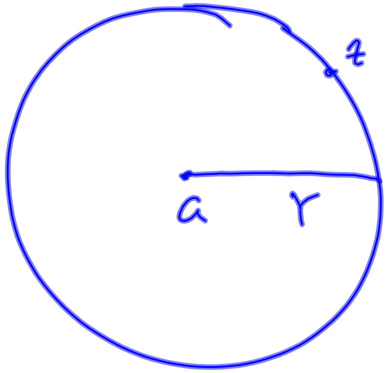
as $r \to \infty$

Corollary of the proof.

If $|f(w)| \leq \text{constant} \cdot |w|^\alpha$ where $\alpha < 1$, then actually f is constant.

Mean-value property of analytic functions in disks

$$f(a) = \frac{1}{2\pi i} \int_{\text{circle}} \frac{f(z)}{z-a}\, dz$$



$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta$$

parametrize circle
by $z = a + re^{i\theta}$
$dz = rie^{i\theta}\, d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

Corollary

$$|f(a)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta})\, d\theta \right|$$

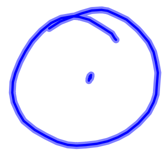
$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{i\theta})|\, d\theta$$

Simplest version of the "maximum principle"

The absolute value of an analytic function cannot have a strict local maximum.

because an average cannot be a strict maximum.

Moreover, $|f|$ can



have a minimum only if that minimum is equal to 0. (otherwise, look at $\frac{1}{f}$ locally.)

January 2015 qualifying examination

8. Suppose f has a simple pole at the origin, and g denotes $1/f$ (the reciprocal function). How is the residue at the origin of the composite function $f \circ g$ related to the residue at the origin of f?

January 2014 qualifying examination

6. Consider a rational function $f(z) = q(z)/p(z)$, where p is a polynomial of degree n and q is a polynomial of degree $n-2$ or less. If $z_1, z_2, ..., z_n$ are distinct roots of p, prove that the residues of f satisfy

$$\sum_{k=1}^{n} Res(f, z_k) = 0.$$

August 2013 qualifying examination

2. Suppose that f is holomorphic in $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$, the punctured unit disk. Prove that the point 0 is a removable singularity for the function f if and only if the point 0 is a removable singularity for the function $f'f''$ (the product of the first derivative of f and the second derivative of f).