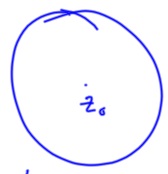
Zeroes of nonconstant analytic functions are isolated

$f(z_0) = 0$

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

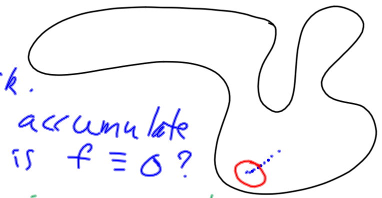


If N is the index of first nonzero coefficient, then

$$f(z) = (z - z_0)^N \underbrace{\left(c_N + c_{N+1}(z - z_0) + \cdots\right)}$$

continuous, nonzero at z_0, so nonzero in some neighborhood.

Suppose f analytic in some open region, not necessarily disk.



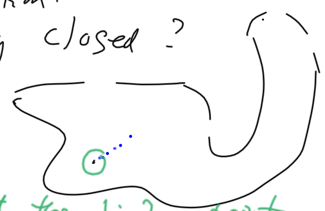
If zeroes of f accumulate somewhere inside, is $f \equiv 0$?

Yes if the region is connected.

Let S be the set of points in the region such that $f \equiv 0$ in a neighborhood of the point.

S is open by definition.

Why is S relatively closed?



By previous proposition, if zeroes of f accumulate, then $f \equiv 0$ in some disk centered at the limit point

S is both open and relatively closed, so either $S = \phi$ or $S =$ the whole connected set.

Identity principle

If f is analytic on a connected open set, and the zeroes of f have an accumulation point in the interior of the set, then f is identically zero.

<u>Corollary</u> If f and g are two analytic functions that agree on a set having an accumulation point inside the region, then $f \equiv g$.

<u>Example</u> Does there exist a function f analytic in the unit disk such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n} \quad (n \geq 2) \quad ?$$

No. Consider $f(z) - z$. This function is 0 at $\frac{1}{2m}$ for every $m \geq 1$. So $f(z) - z$ should be $\equiv 0$. But $f\left(\frac{1}{3}\right) - \frac{1}{3} \neq 0$.

Morera's theorem

converse of Cauchy's theorem

If f is continuous, and for sufficiently many closed curves

$$\int_\gamma f(z)\,dz = 0,$$

then f is analytic.

connected open set

8. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all z in G. Show that either $f \equiv 0$ or $g \equiv 0$.

[Interpretation: the ring of analytic functions is an integral domain.]

10. Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is analytic then either f is constant or $g \equiv 0$.