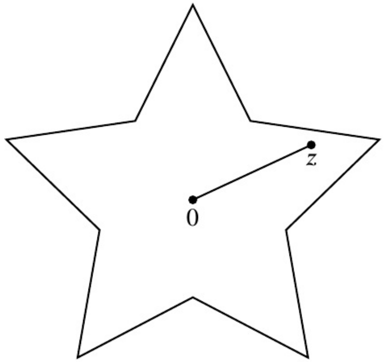
Generalization to starshaped regions

Theorem. *An analytic function in a star-shaped region has an anti-derivative.*



$$F(z) = \int_0^z f(w)\,dw$$

(integrate along a line segment)

$$= \int_0^1 f(tz)\,z\,dt \qquad \text{(parametrize via } w = tz\text{)}$$

Leibniz rule to differentiate inside the integral

$$F'(z) = \int_0^1 \left(f(tz) + f'(tz)\,tz\right)\,dt$$

chain rule
product rule

$$= \int_0^1 \frac{d}{dt}\left(f(tz)\,t\right)\,dt$$

$$= \left[f(tz)\,t\right]_0^1 \quad = f(z)$$

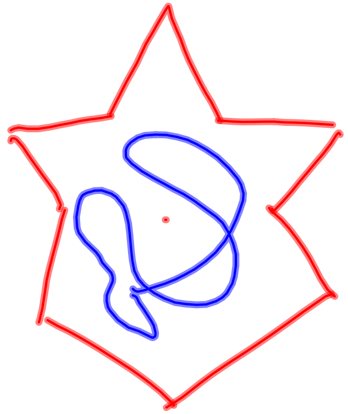
Corollary (Cauchy's integral theorem in star-shaped regions)**.** *If f is an analytic function inside a star-shaped region, and γ is a piecewise continuously differentiable closed curve lying inside the region, then $\int_\gamma f(z)\,dz = 0$.*

$$\int_\gamma f(z)\,dz$$

$$= \int_\gamma F'(z)\,dz$$

$$= F(\text{end point}) - F(\text{start point})$$

$$= 0.$$



Some questions to be answered later

1. For which open sets is $\int_\gamma f(z)\,dz = 0$ for every analytic function f and every closed curve γ in the open set? simply connected open sets

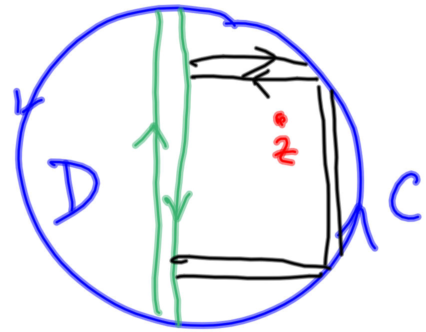
2. Given an open set, what property of a curve γ guarantees that $\int_\gamma f(z)\,dz = 0$ for every analytic function f in the open set? winding number

3. Which functions in a rectangle have the property that $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in the rectangle? Morera's theorem

Going around in circles

Suppose f is analytic in an open set containing a disk D bounded by a circle C.

$$\int_C f(z)\,dz = 0$$

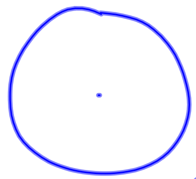


$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = f(z)$$

reduce to the case of a rectangle by adding and subtracting line segments and using the previous theorem for star-shaped regions.

Sample application of the residue theorem for circles

$$\int_0^{2\pi} \frac{1}{5+3\sin\theta}\, d\theta = \frac{\pi}{2}$$



$z = e^{i\theta}$ parametrizes unit circle

$dz = ie^{i\theta}\, d\theta$

$$= \int \frac{1}{5+3\cdot\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)} \frac{dz}{ie^{i\theta}}$$

$$= \int_{\text{circle}} \frac{1}{5+\frac{3}{2i}\left(z-\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \int \frac{1}{\underbrace{5iz+\frac{3}{2}\left(z^2-1\right)}_{\frac{3}{2}(z+3i)(z+i/3)}} dz$$

quadratic formula

Denominator is zero when

$$z = \frac{-5i \pm \sqrt{-25+9}}{3} = -3i,\ -\frac{i}{3}$$

So integral $= 2\pi i \cdot \operatorname{Res}\left(\frac{1}{\text{quadratic}}, -\frac{i}{3}\right)$

$$= 2\pi i \cdot \left.\frac{1}{\frac{3}{2}(z+3i)}\right|_{z=-\frac{i}{3}} = \frac{\pi}{2} \checkmark$$

January 2013 qualifying exam

1. Suppose $\sum_{n=0}^{\infty} a_n z^n$ is the Maclaurin series of the rational function $\dfrac{z}{1-z-z^2}$. Prove that the coefficient sequence $\{a_n\}_{n=0}^{\infty}$ is the sequence of Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ... (defined by the property that each number is the sum of the preceding two).

August 2014 qualifying exam

6. Prove that if $0 < |z| < 1$, then $\frac{1}{4}|z| < |1 - e^z| < \frac{7}{4}|z|$.

(you need to know $e \approx 2.718$)

3. Show that $$\int_0^{2\pi} \frac{\cos(\theta)}{1-\cos(\theta)+\frac{1}{4}}\,\mathrm{d}\theta = \frac{4\pi}{3}.$$