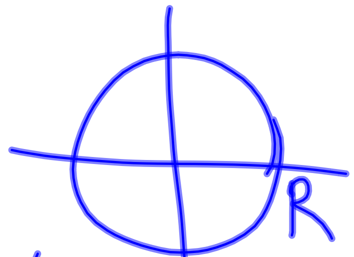
Existence of radius of convergence for $\sum_{n=0}^{\infty} a_n z^n$

Suppose series converges when $z = z_1$.

If now $|z_2| < |z_1|$, then



$$|a_n z_2^n| = |a_n z_1^n| \cdot \left|\frac{z_2}{z_1}\right|^n \leq \text{constant} \cdot \left|\frac{z_2}{z_1}\right|^n$$

where $|a_n z_1^n| \to 0$ so has a constant upper bound.

Set R equal to supremum of $|z_1|$ for all z_1 for which $\sum a_n z_1^n$ converges.

The series converges absolutely and uniformly when $|z| \leq r < R$.

Formula for radius of convergence

By root test, $\sum_n a_n z^n$ converges when

$$\limsup_{n\to\infty} |a_n z^n|^{1/n} < 1$$

diverges when >1

or $$|z| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1.$$

So $$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

Exercises from page 33

6. Find the radius of convergence for each of the following power series: (a) $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$; (b) $\sum_{n=0}^{\infty} a^{n^2} z^n$, $a \in \mathbb{C}$; (c) $\sum_{n=0}^{\infty} k^n z^n$, k an integer $\neq 0$; (d) $\sum_{n=0}^{\infty} z^{n!}$.

if $|a|<1$ then $R=\infty$; if $|a|>1$ then $R=0$; if $|a|=1$, $R=1$

7. Show that the radius of convergence of the power series

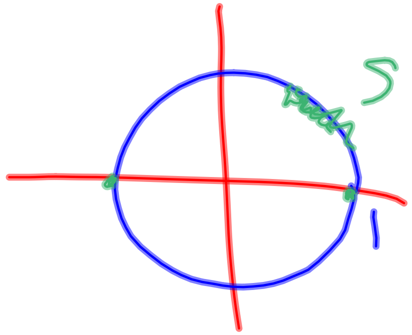
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1$, -1, and i. (Hint: The nth coefficient of this series is not $(-1)^n/n$.)

$z = i$ $\quad (-1)^n \cdot i^{n(n+1)}$ $\quad$ pattern $+1, -1, -1, +1 \ldots$

Open problem (Convergence sets of power series). Call a subset S of $\{ z \in \mathbb{C} : |z| = 1 \}$ (the unit circle) a *convergence set* if there exists a power series with radius of convergence equal to 1 that converges when $z \in S$ and diverges when $|z| = 1$ but $z \notin S$.

<u>Example</u> $\sum_{n=1}^{\infty} z^n$ $\quad S = \emptyset$



$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ $\quad S =$ whole circle

$\sum_{n=1}^{\infty} \frac{z^n}{n}$ $\quad S =$ whole circle $\setminus \{1\}$

Characterize the convergence sets.