Characterization of continuity at a point c

$$f(z_n) \to f(c) \qquad \text{whenever} \qquad z_n \to c.$$

complex

Equivalent definitions of differentiability at a point c

1. $\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$ exists.

2. There is a function g, continuous at c, such that $f(z) - f(c) = g(z)(z - c)$.

3. There is a complex-linear transformation $T : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z \to c} \frac{f(z) - f(c) - T(z - c)}{z - c} = 0.$$

Real differentiability

$F: \mathbb{R}^2 \to \mathbb{R}^2$, say $\qquad F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$

F is differentiable in the real sense at a point $\begin{pmatrix} a \\ b \end{pmatrix}$ if

$\exists$ a real linear transformation T such that

$$\lim_{(x,y)\to(a,b)} \frac{F\begin{pmatrix} x \\ y \end{pmatrix} - F\begin{pmatrix} a \\ b \end{pmatrix} - T\begin{pmatrix} x-a \\ y-b \end{pmatrix}}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

Linear transformation T is the Jacobian matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

If these partial derivatives are continuous, then F is differentiable.

From previous homework, this real-linear T corresponds to a complex-linear transformation when

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equations

Theorem. *A function is complex-differentiable at a point in $\mathbb{C}$ if and only if (the function is real-differentiable at the point and the Cauchy–Riemann equations hold) at the point.*

Theorem. *A sufficient condition for real-differentiability is that the first-order partial derivatives are continuous.*

Theorem A sufficient condition for complex differentiability is that the first order partial derivatives are continuous and satisfy the C-R equations.

Definition. A function defined on an open subset of $\mathbb{C}$ is *analytic* (or *holomorphic*) if the first-order partial derivatives exist, are continuous, and satisfy the Cauchy–Riemann equations.

<u>Examples</u> polynomials in z,

rational functions of z where the open set excludes points where the denominator is 0,

power series where the open set is the open disk where series converges

<u>Examples</u> of real differentiable functions that are not analytic:

$\bar{z} = x - iy$, $|z|^2$

Menshov

Theorem (P. Montel 1913, H. Looman 1923, D. Menchoff 1935)**.** *In the definition of analytic function, the hypothesis of continuity of the partial derivatives can be weakened to continuity of the function.*

Exercise from page 43

1. Show that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

Exercise. Show that the function equal to $z^5/|z|^4$ when $z \neq 0$ and equal to 0 when $z = 0$ is continuous, and the Cauchy–Riemann equations hold at the origin, but the function is not complex-differentiable at the origin.