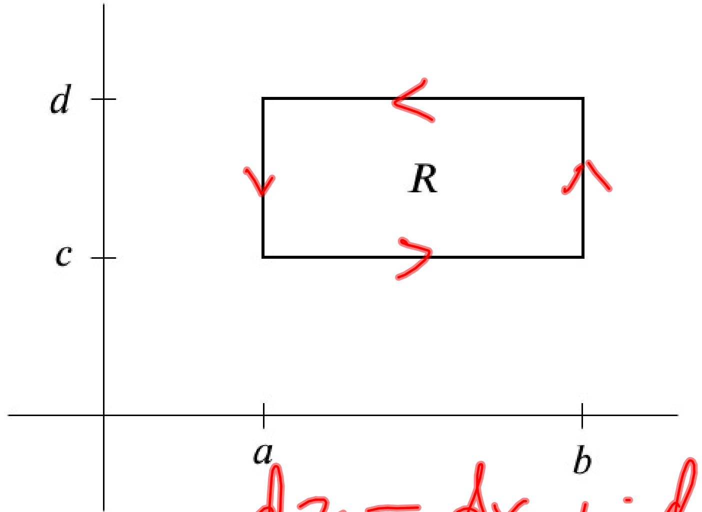
Cauchy's idea of 1814

f analytic



$dz = dx + i\,dy$

$$0 = \iint_R i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\,dx\,dy$$

$$= \int_c^d [f(b,y) - f(a,y)]\,\overset{dz}{\cancel{i\,dy}} + \int_a^b [f(x,c) - f(x,d)]\,\underset{dz}{\cancel{dx}}$$

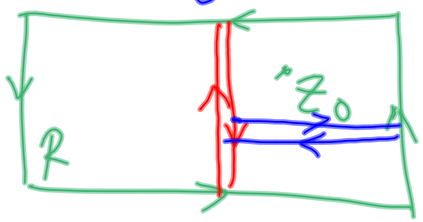
$$= \int_{\partial R} f(z)\,dz$$

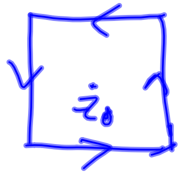
Suppose f is analytic except for one singular point, say

$$f(z) = \frac{g(z)}{z - z_0},$$

where g analytic

Reduce to case of integration around a small square centered at z_0.





$$\int_{\partial(\text{square})} \frac{g(z)}{z - z_0}\, dz$$

$$= \int_{\partial(\text{square})} \left(\frac{g(z) - g(z_0)}{z - z_0} + \frac{g(z_0)}{z - z_0} \right) dz$$

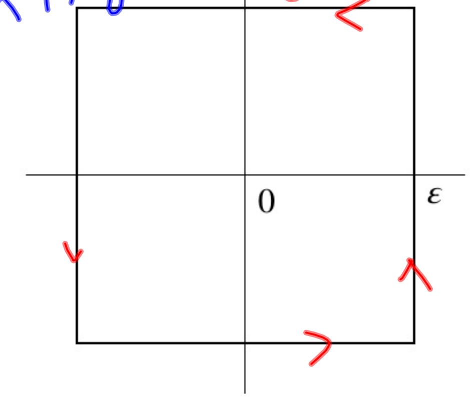
$$= \mathcal{E}(z) + g(z_0) \int_{\partial(\text{square})} \frac{1}{z - z_0}\, dz$$

where $\mathcal{E}(z)$ is small because g is complex differentiable and the integration path is as short as desired.

The calculation reduces to

$$\frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$\int_{\partial(\text{square})} \frac{x-iy}{x^2+y^2} \underbrace{(dx + i\,dy)}_{dz}$$



bottom edge

$$\int_{-\varepsilon}^{\varepsilon} \frac{x+i\varepsilon}{x^2+\varepsilon^2} \quad dx + i\,\cancelto{0}{dy}$$

$$= \underset{\text{by symmetry}}{0} + \int_{-\varepsilon}^{\varepsilon} \frac{i\varepsilon}{x^2+\varepsilon^2}\,dx$$

$$\overset{u=x/\varepsilon}{=} \int_{-1}^{1} \frac{i}{u^2+1}\,du = \frac{\pi}{2} i$$

other three integrals similar,
so final answer is $2\pi i$.

Cauchy's definition of residue (1826)

If $f(z)$ can be expanded near z_0 in a series in powers of $(z-z_0)$ and $1/(z-z_0)$, then the *residue* of f at z_0 is the coefficient of $1/(z-z_0)$.

Theorem (Cauchy's residue theorem for rectangles). *If f is analytic except at isolated points inside a rectangle R, then*

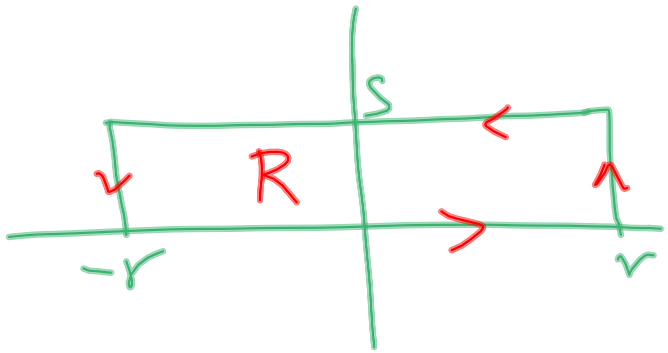
$$\int_{\partial R} f(z)\,dz = 2\pi i \times (\textit{sum of residues at the singular points}).$$

Example

$$\int_0^\infty \frac{\cos(x)}{x^2+1}\,dx$$

$$= \lim_{r\to\infty}\int_0^r \frac{\cos(x)}{x^2+1}\,dx \qquad = \frac{1}{2}\lim_{r\to\infty}\int_{-r}^r \frac{\cos(x)}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\mathrm{Re}\lim_{r\to\infty}\int_{-r}^r \frac{e^{ix}}{x^2+1}\,dx$$



Consider $\lim\limits_{r,s\to\infty} \mathrm{Re}\,\frac{1}{2}\int_{\partial R}\frac{e^{iz}}{z^2+1}\,dz$

$= 2\pi i \times$ residue of $\frac{1}{2}\frac{e^{iz}}{z^2+1}$ at $z=i$

Claim limit of extra integrals over three additional sides $= 0$.

4. Apply the Residue Theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+1)(x^2+4)}.$$