Reminder: Exam 1 takes place Thursday, October 1

Assignment for next time: make a list of the most important concepts and theorems covered so far.

Theorem (Cauchy's residue theorem for rectangles)**.** *If f is analytic except at isolated points inside a rectangle R, then*

$$\int_{\partial R} f(z)\,dz = 2\pi i \times (\textit{sum of residues at the singular points}).$$

(proved so far only for first-order singularities)

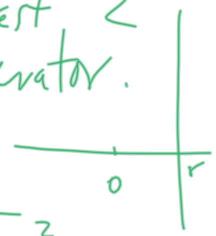
Assume finitely many singularities.

Theorem (after Cauchy)**.** *If $f(x+yi)$ tends to 0 uniformly with respect to x when $y \to \infty$, and if $\int_0^\infty |f(x+yi)|\,dy$ tends to 0 when $x \to \pm\infty$, then $\int_{-\infty}^{\infty} f(x)\,dx$ equals $2\pi i$ times the sum of the residues of f in the upper half-plane.*

$\forall \varepsilon > 0 \quad \exists A$ independent of x such that

$|f(x+yi)| < \varepsilon$ when $y > A$

Example: Rational function with degree of denominator at least 2 bigger than degree of numerator.



On vertical line $r+iy$,

$$|f(r+iy)| \le C \cdot \frac{1}{|r+iy|^2} = C \cdot \frac{1}{r^2+y^2}$$

and

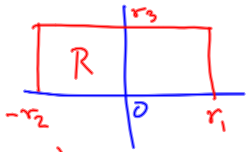
$$\int_0^\infty \frac{1}{r^2+y^2}\,dy = \frac{1}{r}\tan^{-1}\frac{y}{r}\Big|_0^\infty$$

$$= \frac{\pi}{2}\cdot\frac{1}{r} \to 0 \text{ when } r \to \infty.$$

Example 2: $f(z) = e^{iz}$ times a rational function with denominator at least 1 greater than degree of numerator.

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = e^{-y}$$

Proof of theorem.



$$\int_{\partial R} f(z)\,dz = 2\pi i \times (\text{sum of residues})$$

$$= \int_{-r_2}^{r_1} f(x)\,dx + \int_{\sqcap} f(z)\,dz$$

Want to show

$$\lim \int_{\sqcap} f(z)\,dz = 0$$

Fix $\varepsilon > 0$. Choose R_1 and R_2 such that

$$\int_0^\infty |f(x+iy)|\,dy < \frac{\varepsilon}{3}$$

when $x > R_1$ and when $x < -R_2$.

Now if $r_1 > R_1$ and $r_2 > R_2$, then choose R_3 such that

$$|f(x+iy)| \leq \frac{\varepsilon}{3(r_1+r_2)}$$

when $y \geq R_3$.

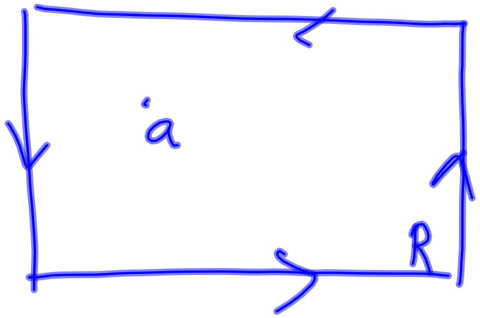
Then

$$\left| \int_{\sqcap\,(-r_2,\,r_1,\,r_3)} f(z)\,dz \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon \cdot (r_1+r_2)}{3(r_1+r_2)}$$

$$= \varepsilon$$

Consequences of the rectangle theorem

Suppose f analytic in R



$$\frac{1}{2\pi i}\int_{\partial R}\frac{f(w)}{w-a}\,dw = f(a)$$

Cauchy's integral theorem
or integral representation

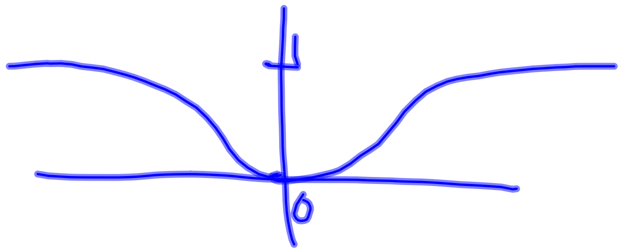
$$\frac{1}{2\pi i}\int_{\partial R}\frac{f(w)}{(w-a)^2}\,dw = f'(a)$$

by Leibniz's rule
for differentiating
an integral

$$\frac{n!}{2\pi i}\int_{\partial R}\frac{f(w)}{(w-a)^{n+1}}\,dw = f^{(n)}(a)$$

Cauchy's example of a real function that is not represented by its Maclaurin series

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$f^{(n)}(0) = 0$ for every n

$f(x) \neq$ its Maclaurin series when $x \neq 0$