Examination 1

1. State some theorem (from this course) to which the name of Augustin-Louis Cauchy is attached.
2. Give a geometric description of the set of points z in $\mathbb{C}$ for which
$$z^2 + 4z\overline{z} + (\,\overline{z}\,)^2 = 6.$$
3. The complex function $\tan(z)$ is defined to be the quotient $\dfrac{\sin(z)}{\cos(z)}$. Show that there is no complex number z for which $\tan(z)$ is equal to i.
4. Suppose the power series $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence equal to 6, and the power series $\sum_{n=1}^{\infty} b_n z^n$ has radius of convergence equal to 7. What, if anything, can be said about the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n b_n z^n$?
5. Suppose f is an analytic function (on some open subset of $\mathbb{C}$) with real part u and imaginary part v. Show that ∇u and ∇v are orthogonal vectors. The notation ∇u means $\left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$, the gradient vector of u.
6. Show that
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{3}.$$

Bonus

Who is the French mathematician shown in the picture below?

(1789–1857)