Examination 1

1. State some theorem (from this course) to which the name of Augustin-Louis Cauchy is attached.

   Solution. Some results covered so far are Cauchy's integral theorem for rectangles, Cauchy's residue theorem for rectangles, and Cauchy's formula for the radius of convergence of a power series. (There are others.)

2. Give a geometric description of the set of points z in $\mathbb{C}$ for which

   $$z^2 + 4z\overline{z} + (\,\overline{z}\,)^2 = 6.$$

   Solution. The equation is quadratic, so the set must be some conic section. To identify the set more precisely, you could set z equal to $x+iy$ and rewrite the left-hand side in terms of the real coordinates x and y:

   $$\begin{aligned} z^2 + 4z\overline{z} + (\,\overline{z}\,)^2 &= 2\,\mathrm{Re}(z^2) + 4|z|^2 \\ &= 2(x^2 - y^2) + 4(x^2 + y^2) \\ &= 6x^2 + 2y^2. \end{aligned}$$

   Accordingly, the original equation says that $x^2 + \frac{1}{3}y^2 = 1$. This equation represents an ellipse centered at the origin. (The foci are $\pm\sqrt{2}\,i$.)

3. The complex function $\tan(z)$ is defined to be the quotient $\dfrac{\sin(z)}{\cos(z)}$. Show that there is no complex number z for which $\tan(z)$ is equal to i.

   Solution. Seeking a contradiction, suppose that there is a complex number z for which $\tan(z) = i$, or, equivalently, $\sin(z) = i\cos(z)$. Multiplying by $-i$ shows that $-i\sin(z) = \cos(z)$, or $0 = \cos(z)+i\sin(z) = e^{iz}$. But the complex exponential function is never equal to 0: if $z = x + iy$, then $|e^{iz}| = e^{-y} > 0$. The contradiction shows that $\tan(z)$ is never equal to i.

   An alternative method is to start from the supposition that $\sin(z) = i\cos(z)$ and square both sides to deduce that $\sin^2(z) = -\cos^2(z)$, whence $\sin^2(z) + \cos^2(z) = 0$. But $\sin^2(z) + \cos^2(z) = 1$ for every complex number z by the persistence of functional relationships, since $\sin^2(x)+\cos^2(x) = 1$ when x is a real number. The contradiction again shows that the point i cannot be in the range of the complex tangent function.

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Remark. Since the tangent function is odd (antisymmetric), the tangent function cannot take the value $-i$ either. The two values $\pm i$ turn out to be the only values omitted by the complex tangent function. You can verify this claim by an explicit calculation; alternatively, the claim can be deduced from a deep theorem of Picard, to be proved in Math 618, concerning omitted values of meromorphic functions.

4. Suppose the power series $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence equal to 6, and the power series $\sum_{n=1}^{\infty} b_n z^n$ has radius of convergence equal to 7. What, if anything, can be said about the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n b_n z^n$?

Solution. The hypothesis implies that $\limsup_{n\to\infty} |a_n|^{1/n} = 1/6$, and similarly $\limsup_{n\to\infty} |b_n|^{1/n} = 1/7$. The lemma stated and proved below implies that $\limsup_{n\to\infty} |a_n b_n|^{1/n} \le 1/42$. Therefore the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n b_n z^n$ is at least 42 but could be larger.

Indeed, if r is an arbitrary number greater than or equal to 42, then there is an example in which the radius of convergence of $\sum_{n=1}^{\infty} a_n b_n z^n$ is equal to r. Namely, set a_n equal to $1/6^n$ when n is a power of 2 and 0 otherwise; set b_n equal to $(6/r)^n$ when n is a power of 2 and $1/7^n$ when n is a power of 3 and 0 otherwise. Evidently, $\limsup_{n\to\infty} |a_n|^{1/n} = 1/6$; and $6/r \le 1/7$ when $r \ge 42$, so $\limsup_{n\to\infty} |b_n|^{1/n} = 1/7$. The value of $a_n b_n$ equals $1/r^n$ when n is a power of 2 and 0 otherwise, so $\limsup_{n\to\infty} |a_n b_n|^{1/n} = 1/r$, as desired. The special case when $r = \infty$ can be handled by the same argument by interpreting $6/r$ as 0 in that case.

Lemma. Suppose for every natural number n that $A_n \ge 0$ and $B_n \ge 0$. Let α denote $\limsup_{n\to\infty} A_n$, and let β denote $\limsup_{n\to\infty} B_n$. If α and β are finite (not $+\infty$), then

$$\limsup_{n\to\infty} A_n B_n \le \alpha\beta.$$

Remark. The reason for excluding an infinite lim sup is that if $\alpha = \infty$ and $\beta = 0$, then the right-hand side of the inequality is the undefined expression $\infty \cdot 0$.

Proof. Fix an arbitrary positive number ε. The definitions of α and β imply the existence of a number N such that $A_n \le \alpha + \varepsilon$ and $B_n \le \beta + \varepsilon$ whenever $n \ge N$. The indicated quantities are nonnegative, so $A_n B_n \le (\alpha + \varepsilon)(\beta + \varepsilon)$

when $n \geq N$. Therefore

$$\limsup_{n\to\infty} A_n B_n \leq (\alpha + \varepsilon)(\beta + \varepsilon).$$

Now let ε tend to 0.

5. Suppose f is an analytic function (on some open subset of $\mathbb{C}$) with real part u and imaginary part v. Show that ∇u and ∇v are orthogonal vectors. The notation ∇u means $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$, the gradient vector of u.

   Solution. The dot product (scalar product) of the two gradients equals

   $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}.$$

   By the Cauchy–Riemann equations, this expression equals

   $$\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}, \qquad \text{or} \qquad 0.$$

   Accordingly, the two vectors are orthogonal.

   Remark. The geometric interpretation is that the level sets of u (the curves where u has a constant value) and the level sets of v are families of mutually perpendicular curves (which you might have called "orthogonal trajectories" in calculus class).

6. Show that
   $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}.$$

   Solution. The integrand is a rational function with the degree of the denominator two greater than the degree of the numerator. By the corollary of Cauchy's residue theorem for rectangles discussed in class, the value of the integral is $2\pi i$ times the sum of the residues of the rational function $\dfrac{z^2}{(z^2+1)(z^2+4)}$ at the two singularities in the upper half-plane: namely, the point where $z = i$ and the point where $z = 2i$.

Since

$$\frac{z^2}{(z^2+1)(z^2+4)} = \frac{\frac{z^2}{(z+i)(z^2+4)}}{z-i},$$

the residue at the point where $z = i$ is equal to

$$\left.\frac{z^2}{(z+i)(z^2+4)}\right|_{z=i}, \qquad \text{or} \qquad \frac{-1}{6i}.$$

Similarly, the residue at the point where $z = 2i$ is equal to

$$\left.\frac{z^2}{(z^2+1)(z+2i)}\right|_{z=2i}, \qquad \text{or} \qquad \frac{-4}{-12i}.$$

The sum of these residues equals $1/(6i)$, so the value of the integral equals $2\pi i/(6i)$, or $\pi/3$.

Remark. This integral can be computed instead by techniques of real calculus using the method of partial fractions: namely,

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-1/3}{x^2+1} + \frac{4/3}{x^2+4}.$$

The two fractions on the right-hand side have elementary antiderivatives in terms of the real arctangent function. Cauchy's residue calculus can be viewed in this instance as a shortcut that eliminates some of the algebra involved in computing the partial-fractions decomposition.

Bonus

Who is the French mathematician shown in the picture below?

(1789–1857)

Solution. This mathematician is Augustin-Louis Cauchy, the person who created most of the mathematics under discussion in this course.