Final Examination

Instructions Solve *six* of the following eight problems (most of which are taken from the textbook).

1. State the following theorems (with precise hypotheses and conclusions): Riemann's theorem on removable singularities, the homology version of Cauchy's theorem, and one of Picard's theorems.

   Solution. One version of Riemann's theorem says that if an analytic function is bounded in a punctured neighborhood of an isolated singularity, then the singularity is removable; a more sophisticated version is Theorem 1.2 in §V.1 of the textbook. The homology version of Cauchy's theorem says that if f is analytic in an open set G, and γ is a closed curve in G that has winding number zero about each point in the complement of G, then $\int_\gamma f(z)\,dz = 0$. Picard's great theorem (which we did not prove) says that in every punctured neighborhood of an essential singularity, an analytic function assumes every complex value infinitely often, with one possible exceptional value. Picard's little theorem states that the same conclusion holds for nonpolynomial entire functions.

2. Find both the real part and the imaginary part of $\left(\dfrac{-1-i\sqrt{3}}{2}\right)^6$.

   Solution. This problem is a part of Exercise 1 in §I.2. One method is to recognize $\dfrac{-1-i\sqrt{3}}{2}$ as being $-e^{\pi i/3}$. Therefore the sixth power equals $e^{2\pi i}$, or 1. So the real part is 1 and the imaginary part is 0.

3. Let G be the complex plane slit along the negative part of the real axis: namely, $G = \mathbb{C} \setminus \{\, z \in \mathbb{R} : z \le 0 \,\}$. Let n be a positive integer. There exists an analytic function $f : G \to \mathbb{C}$ such that $z = (f(z))^n$ for every z in G. Find every possible such function f.

   Solution. This item is Exercise 13 in §III.2. The question asks for all analytic nth roots of the identity function in the slit plane. One such function is $\exp\left(\frac{1}{n}\log(z)\right)$, where $\log(z)$ means the principal branch of the logarithm [that is, $\log(z) = \ln|z| + i\arg(z)$, where $-\pi < \arg(z) < \pi$].

If f_1 and f_2 are two nth roots of z, then $\left(f_1(z)/f_2(z)\right)^n$ is identically equal to 1. Accordingly, there is an integer k such that $0 \le k \le n-1$ and

$$\frac{f_1(z)}{f_2(z)} = e^{2\pi ik/n}.$$

Therefore every nth root of z can be written in the form

$$e^{2\pi ik/n} \exp\left(\tfrac{1}{n}\log(z)\right)$$

for some k between 0 and $n-1$. If z is written in polar coordinates as $re^{i\theta}$, where $r > 0$, and $-\pi < \theta < \pi$, then

$$f(z) = r^{1/n} \exp\left(\frac{i\theta}{n} + \frac{2\pi ik}{n}\right), \qquad \text{where } 0 \le k \le n-1.$$

4. Evaluate $\displaystyle\int_\gamma \frac{e^{iz}}{z^2}\,dz$ when γ is the unit circle traversed once counterclockwise [that is, $\gamma(t) = e^{it}$ and $0 \le t \le 2\pi$].

   Solution. This item is Exercise 7(a) in §IV.2. By Cauchy's integral formula for the first derivative, this integral equals $2\pi i$ times the value at the origin of the derivative of e^{iz}. Thus the answer is $(2\pi i)ie^0$, or -2π.

5. Let r be a positive real number, let θ be a real number, and let γ be a continuously differentiable curve in $\mathbb{C} \setminus \{0\}$ that begins at the point 1 and ends at the point $re^{i\theta}$. Prove the existence of an integer k such that

$$\int_\gamma \frac{1}{z}\,dz = \ln(r) + i\theta + 2\pi ik.$$

   Solution. This item is Exercise 4 in §IV.4. One solution is essentially the same as the proof that the winding number of a closed curve about a point is always an integer, except that here the curve is not closed.

   Suppose the curve γ is parametrized from 0 to 1: thus $\gamma(0) = 1$, and $\gamma(1) = re^{i\theta}$. Let g be the function defined on the unit interval defined as follows:

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds.$$

Then $g(0) = 0$, and $g(1) = \int_\gamma \frac{1}{z}\,dz$. The goal is to show that $e^{g(1)} = \gamma(1)$. Consider the function $e^{-g(t)}\gamma(t)$. The derivative equals

$$e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}\gamma(t),$$

which reduces to 0 (because $g'(t) = \gamma'(t)/\gamma(t)$ by the fundamental theorem of calculus). Therefore $e^{-g(t)}\gamma(t)$ is a constant function of t. The constant value equals $e^{-g(0)}\gamma(0)$, or 1. Therefore $e^{g(1)} = \gamma(1)$, as required.

An alternative solution is to apply the knowledge that the winding number of a closed curve about a point is necessarily an integer. Let σ be a simple curve (no self-intersection) from 1 to $re^{i\theta}$. The curve σ has a simply connected neighborhood not containing 0. A branch of the logarithm can be defined in this neighborhood, and

$$\int_\sigma \frac{1}{z}\,dz = \log(re^{i\theta}) - \log(1) = \ln(r) + i\theta + 2\pi i j$$

for some integer j that depends on the chosen branch of the logarithm. Now the difference $\displaystyle\int_\gamma \frac{1}{z}\,dz - \int_\sigma \frac{1}{z}\,dz$ equals the integral of $\dfrac{1}{z}$ over a closed curve (namely, γ followed by σ in reverse), hence equals $2\pi i\ell$ for some integer ℓ. The required integer k is $j + \ell$.

6. Suppose $f(z) = \dfrac{1}{1 - e^z}$. Find the singular part of f at each singular point.

Solution. This item is Exercise 1(h) in §V.1. The singular points are the values of z for which $e^z = 1$: namely, $2\pi i n$ for an arbitrary integer n. The derivative of $1 - e^z$ is never equal to 0, so each zero of the denominator is simple. Thus f has only simple poles.

The singular part (or principal part) of $f(z)$ at $2\pi i n$ therefore has the form $\dfrac{c_n}{z - 2\pi i n}$, where c_n is the residue at the singular point. This residue equals

$$\lim_{z\to 2\pi i n} \frac{z - 2\pi i n}{1 - e^z}, \qquad \text{or} \qquad \left.\frac{1}{\frac{d}{dz}(1 - e^z)}\right|_{z=2\pi i n}, \qquad \text{or} \qquad -1.$$

Accordingly, the singular part of $f(z)$ at $2\pi i n$ is $-1/(z - 2\pi i n)$.

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7. Let G be a bounded open set. Suppose the function f is continuous on the closure of G and analytic on G. Additionally, suppose there is a positive constant c such that $|f(z)| = c$ for every z on the boundary of G. Prove that either f is a constant function, or f has at least one zero in G.

   Solution. This item is Exercise 2 in §VI.1. There should be an additional hypothesis that the set G is connected.

   By the maximum principle, $|f(z)| \leq c$ for every point z inside G. If f has no zero in G, then $1/f$ is analytic, and $|1/f(z)| = 1/c$ for every z on the boundary of G. Again by the maximum principle, $1/|f(z)| \leq 1/c$ for every z in G; equivalently, $|f(z)| \geq c$ for every z in G. Combining this inequality with the previous one shows that $|f(z)| = c$ for every z in G. Thus the range of f lies on the boundary of a disk of radius c. By the open-mapping theorem, the function f reduces to a constant function. In other words, if f has no zero, then f is a constant function.

   The example of G equal to the unit disk and $f(z)$ equal to z^2 shows that non-constant analytic functions with constant absolute value on the boundary do exist. There is an extensive theory of such functions, called *inner functions*.

8. Apply Rouché's theorem to prove the fundamental theorem of algebra (every polynomial of degree n has n zeroes, counting multiplicity).

   Solution. This application is worked out in the textbook at the end of §V.3. The idea is that on every sufficiently large circle, a polynomial is a small perturbation of the leading term (small in the sense of Rouché's theorem). Therefore a polynomial has the same number of zeroes in the plane as the leading term does.