Math 617 **Exam 1** **7 October 1994**

1. (a) Show that if $\{z_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $\mathrm{Re} z_n \to \infty$, then $z_n \to \infty$. (Recall that $z_n \to \infty$ means that $d(z_n, \infty) \to 0$, where d is the spherical metric.)
   (b) Is the converse true? That is, if $z_n \to \infty$, must it be the case that $\mathrm{Re} z_n \to \infty$? Prove or give a counterexample.

2. For each part, give an example of a nonempty, proper subset of the metric space $\mathbb{C}$ (with the usual Euclidean metric) satisfying the given properties.
   (a) closed but not compact
   (b) open, dense, and disconnected

3. Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{2+z^n}$.
   Warning: this is not a power series! Therefore the theorem you know about the radius of convergence is not directly applicable.
   (a) Find the largest open set of points z in $\mathbb{C}$ (with the usual metric) for which the series converges.
   (b) Show that the series converges uniformly on compact subsets of the open set determined in part (a).
   (c) The series represents an analytic function on the open set determined above. Find the derivative of this function at the point $z = 0$.

4. Find a one-to-one conformal mapping of the open upper half-disk $\{ z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im} z > 0 \}$ onto the open upper half-plane $\{ z \in \mathbb{C} : \mathrm{Im} z > 0 \}$.
   Hint: construct the mapping as a composite of simple mappings. To get started, map to an unbounded domain by sending a suitable boundary point to infinity.