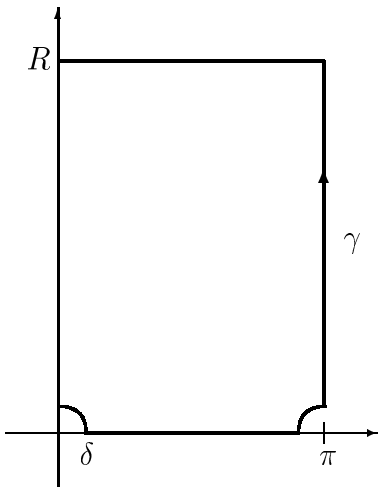
Math 617 Exam 2 21 November 1994

1. (a) Give an example of an open set G in the complex plane, a simple closed differentiable curve γ in G, and a function f that is analytic in G with the property that $\int_\gamma f(z)\,dz \neq 0$. Explain why this example does not contradict Cauchy's theorem.

   (b) Now find an example as above with the additional property that for every positive integer n, the integral $\int_\gamma (f(z))^n\,dz$ is different from zero. Explain why your example works.

2. Let γ be the simple closed curve defined by $\gamma(t) = e^{2\pi i t}$ for $0 \leq t \leq 1$, and let f be an entire function. Suppose that for each point a in the unit disk (that is, $|a| < 1$), there exists a positive integer $n(a)$ such that

$$\int_\gamma \frac{f(z)}{(z-a)^{n(a)}}\,dz = 0.$$

   Prove that f must be a polynomial.
   Hint. To get started, consider the case when $n(a)$ is independent of the point a. Then do the general case in which the integer depends on the point.

3. Prove that $\displaystyle\int_0^\pi \ln(\sin(x))\,dx = -\pi \ln 2$.
   Remark. Mathematica can compute this convergent improper integral in closed form, but Maple cannot.
   Hint. Look at the integral $\int_\gamma \log(1 - e^{2iz})\,dz$, where log denotes the principal branch of the logarithm, and the curve γ is the one shown in the diagram.