Math 617 Final Exam 13 December 1994

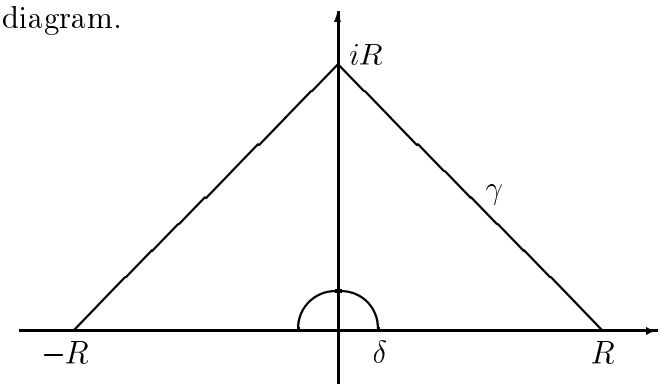
1. Define a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers recursively by

$$\begin{cases} a_0 = -1, \\ a_{n+1} = \dfrac{1}{i + a_n}, \quad n \geq 0. \end{cases}$$

   Find $\sup_n |a_n|$.

2. (a) State Cauchy's integral formula for analytic functions in a disk.
   (b) Use Cauchy's integral formula to prove that an analytic function in a disk can be represented in the disk by a convergent power series.

3. (a) Define Möbius transformations (linear fractional transformations).
   (b) Prove that linear fractional transformations are *continuous* mappings from the extended complex plane $\mathbb{C}_\infty$ onto itself (continuous with respect to the spherical metric).

4. (a) State some version of the maximum modulus theorem for analytic functions.
   (b) Find examples of
      i. an open set G, a nonconstant analytic function f in G, and a point a in G such that $|f(a)| = \sup\{\, |f(z)| : z \in G \,\}$;
      ii. an open set G, a nonconstant analytic function f in G that is continuous on the closure G^-, and a point a in G such that $|f(a)| > \sup\{\, |f(z)| : z \in \partial G \,\}$.

5. Use complex contour integrals to prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.
   Suggestion: integrate a suitable function over the contour γ indicated in the diagram.