Exercise on order and series coefficients

An entire function has a Taylor series expansion $\sum_{n=0}^{\infty} a_n z^n$ that converges in the entire complex plane. Since the entire function is determined by its Taylor series coefficients a_n, its order λ should be computable, in principle, from the a_n.

A formula for λ in terms of the Taylor series coefficients makes it easy to construct examples of entire functions with prescribed orders. Your task is to prove that the order λ is equal to

$$\limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|a_n|}}.$$

Let's temporarily denote this quantity by β; then we want to show (a) $\beta \le \lambda$ and (b) $\lambda \le \beta$.

1. Make sense of the definition of β when some of the coefficients a_n are zero. (A gap series, for instance, has infinitely many of its coefficients equal to zero.)

2. Fix a positive ϵ. By the definition of order, $M(r) < e^{r^{\lambda+\epsilon}}$ for sufficiently large r. Bound $|a_n|$ for large n by applying Cauchy's estimate with $r \approx n^{\frac{1}{\lambda+\epsilon}}$ and deduce that $\beta \le \lambda + \epsilon$. Let $\epsilon \downarrow 0$.

3. Fix a positive ϵ. Then $|a_n| < n^{-\frac{n}{\beta+\epsilon}}$ for sufficiently large n, by the definition of β. Observe that $M(r) \le \sum_{n=0}^{\infty} |a_n| r^n$. By splitting the sum where $n \approx (2r)^{\beta+\epsilon}$, show that $\lambda < \beta + 2\epsilon$. Let $\epsilon \downarrow 0$.