Exercise on Mergelyan's theorem

The goal of this exercise is to demonstrate understanding of the hypotheses of Mergelyan's approximation theorem by explicating appropriate examples.

It is a familiar idea to approximate a complicated function by simpler ones, for instance, by polynomials. In a course on real analysis, you learned the approximation theorem of Weierstrass: every continuous function on a closed interval $[a, b]$ can be approximated uniformly by polynomials.

1. View the closed interval $[a, b]$ as sitting inside the complex plane. Can every continuous function f on $[a, b]$ be approximated uniformly by *holomorphic* polynomials (that is, by polynomials in the complex variable z)? Does it matter whether f is real valued or complex valued?

Although it is not true that an arbitrary continuous function on an arbitrary compact set in $\mathbb{C}$ can be approximated uniformly by holomorphic polynomials, the obstructions to approximability are known. One obstruction is analytic in nature, while the other obstruction is topological.

2. If f is the uniform limit of holomorphic polynomials on a compact set K, then Theorem 3.5.1 on page 90 implies that f must be holomorphic in the *interior* of K.

What is the simplest example you can find of a continuous function on the closed disk $\{ z \in \mathbb{C} : |z| \leq 1 \}$ that cannot be uniformly approximated on the closed disk by holomorphic polynomials?

3. If a compact set has holes, then a function f cannot be approximated by holomorphic polynomials if f has singularities hiding in the holes.

Show that the function f defined by $f(z) = \bar{z}$ cannot be approximated uniformly by holomorphic polynomials on the compact set $\{ z \in \mathbb{C} : |z| = 1 \}$ (which, by the way, is a set with empty interior).

The simplest version of Mergelyan's theorem says that if K is a compact subset of $\mathbb{C}$ having no holes (that is, the complement $\mathbb{C} \setminus K$ is connected), and if f is continuous on K and holomorphic in the interior of K, then f can indeed be approximated uniformly on K by holomorphic polynomials.

An older and weaker theorem of Runge has the same conclusion, but makes the stronger hypothesis that f is holomorphic in an open neighborhood of K (not just in the interior of K).