

Exercise on the $\bar{\partial}$ -equation

The goal of this exercise is to understand how to solve the $\bar{\partial}$ -equation on planar domains.

You know from studying linear algebra and ordinary differential equations that homogeneous equations (ones with zero right-hand side) and inhomogeneous equations (ones with non-zero right-hand side) are intimately related. The same principle applies in the context of holomorphic functions. The fundamental homogeneous equation is $\partial f/\partial\bar{z} = 0$, the short-hand form of the Cauchy-Riemann equations. It is natural to consider the *inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial f}{\partial\bar{z}} = g, \quad (1)$$

where g is given and f is to be found. This is known as the $\bar{\partial}$ -equation.

A typical application of the solvability of this equation is the construction of holomorphic functions with specified properties. First one constructs a class C^1 function (not holomorphic) having the desired properties. Then one subtracts a correction function whose \bar{z} derivative is prescribed to make the new function holomorphic. Of course, one has to make sure that the correction does not destroy the special properties that were attained in the first step.

Our goal here is to see how to solve the $\bar{\partial}$ -equation via an integral formula. The claim is that if g has a continuous derivative in the closure of a bounded region G in the complex plane, then

$$f(z) := -\frac{1}{\pi} \iint_G \frac{g(w)}{w-z} du dv, \quad z \in G, \quad w = u + iv, \quad (2)$$

solves the equation $\partial f/\partial\bar{z} = g$ in G .

To verify that (2) does solve the $\bar{\partial}$ -equation (1), one can apply the Cauchy integral formula with remainder for non-holomorphic functions. That formula says that if g is a class C^1 function on the closure of a bounded domain Ω in \mathbb{C} with class C^1 boundary, then

$$g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{g(w)}{w-z} dw - \frac{1}{\pi} \iint_{\Omega} \frac{\partial g/\partial\bar{w}}{w-z} du dv, \quad z \in \Omega. \quad (3)$$

For a proof of (3) using Green's theorem, see pages 486–487 in Appendix A.

In this exercise, we will assume formula (3) and use it to show that the integral formula (2) does solve the $\bar{\partial}$ -equation (1).

1. Consider the special case that g has *compact support* in G . (This means that g is equal to 0 near the boundary of G ; more formally, the closure of the set of points where g is different from 0 is a compact set in G .) Deduce from (3) that

$$g(z) = -\frac{1}{\pi} \iint_G \frac{\partial g / \partial \bar{w}}{w - z} du dv, \quad z \in G.$$

2. By changing variables in (2) and differentiating under the integral sign, show that

$$\frac{\partial f}{\partial \bar{z}} = -\frac{1}{\pi} \iint_G \frac{\partial g / \partial \bar{w}}{w - z} du dv, \quad z \in G,$$

and deduce that (2) does solve the $\bar{\partial}$ -equation (1) when g has compact support in G .

The result for arbitrary g follows from the case of compactly supported g by the following trick. Fix a point z_0 in G , and let φ be a differentiable bump function that is equal to 1 in a neighborhood of z_0 and equal to 0 outside a larger neighborhood of z_0 . Rewrite (2) as

$$f(z) = -\frac{1}{\pi} \iint_G \frac{g(w)\varphi(w)}{w - z} du dv - \frac{1}{\pi} \iint_G \frac{g(w)(1 - \varphi(w))}{w - z} du dv. \quad (4)$$

3. The \bar{z} derivative of the second integral in (4) equals 0 for z in a neighborhood of z_0 because one can differentiate under the integral sign. (Why does the same reasoning not apply to the first integral?)
4. By the already proved case of compact support, the first integral in (4) has \bar{z} derivative equal to $g(z)$ for z near z_0 . Consequently, (2) solves the $\bar{\partial}$ -equation (1) whether or not g has compact support.