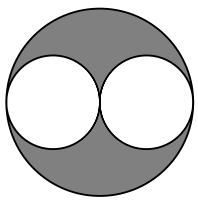
Exercise on Runge's theorem

The goal of this exercise is to demonstrate understanding of Runge's theorem: *If K is a compact subset of $\mathbb{C}$, then every function holomorphic in a neighborhood of K can be approximated uniformly on K by rational functions.* Moreover, if E is a set containing at least one point from each bounded component of $\mathbb{C} \setminus K$, then the poles may be taken to lie in E.


Carl Runge
(1856–1927)



Consider the example illustrated in the figure: K is the set of points z in $\mathbb{C}$ such that $|z| \leq 2$ and $|z-1| \geq 1$ and $|z+1| \geq 1$.

1. Can every function holomorphic in a neighborhood of K be approximated uniformly on K by rational functions with poles only at the points -1 and 1?

2. Give an example of a function holomorphic in a neighborhood of K that cannot be approximated uniformly on K by polynomials but that can be approximated uniformly on K by rational functions with poles only at the point -1.

3. The *polynomially convex hull* $\widehat{K}$ of a compact set K is the set of points that cannot be "separated" from K by a polynomial: namely, if $\|p\|_K$ denotes $\sup\{\, |p(z)| : z \in K \,\}$, then

$$\widehat{K} = \{\, w \in \mathbb{C} : |p(w)| \leq \|p\|_K \text{ for every polynomial } p \,\}.$$

Determine the polynomially convex hull of the compact set K shown in the figure.

4. Let G be the interior of the compact set K shown in the figure; that is, $G = K \setminus \partial K$. The open set G has how many connected components? The complement $\mathbb{C} \setminus G$ has how many connected components?

5. Can every holomorphic function on G be approximated normally on G by polynomials?