

Exercise on Alice Roth's Swiss cheese

If K is a compact set in \mathbb{C} , and f is continuous on K and holomorphic in the interior of K , can f be approximated uniformly on K by rational functions?

S. N. Mergelyan (С. Н. Мергелян) proved that the answer is “yes” when the diameters of the components of the complement of K are bounded away from 0 (in particular, when $\mathbb{C} \setminus K$ has a finite number of components). Runge's theorem implies that the answer is “yes” when f can be approximated uniformly on K by functions holomorphic in an open neighborhood of K . In general, however, the answer is “no”.

The Swiss mathematician Alice Roth constructed a counterexample by punching an infinite number of holes in a disc.¹ Such a set is now called a “Swiss cheese” (which is the generic name in English for a pale-yellow cheese having many holes). Your task is to recreate a version of Alice Roth's example, as follows. (See also starred exercise 18 on page 381 of the textbook.)

1. Demonstrate the existence of a sequence $\{D_j\}_{j=1}^{\infty}$ of open discs satisfying all three of the following properties simultaneously.
 - (a) The closures of the discs D_j are pairwise disjoint and are contained in the open unit disc D .
 - (b) The sum of the radii of the discs D_j is less than $1/2$.
 - (c) The set $K := \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ is compact and has empty interior.
2. Use Cauchy's theorem to show that if R is a rational function whose poles lie in the complement of K , then

$$\oint_{\partial D} R(z) dz = \sum_{j=1}^{\infty} \oint_{\partial D_j} R(z) dz.$$

3. Observe that $\oint_{\partial D} \bar{z} dz = 2\pi i$, while $\left| \sum_{j=1}^{\infty} \oint_{\partial D_j} \bar{z} dz \right| < \pi$.
4. Deduce that the function f defined by $f(z) = \bar{z}$ is an example of a continuous function on K that is holomorphic in the interior of K (vacuously, since K has empty interior) and that cannot be arbitrarily well approximated uniformly on K by rational functions.

¹Alice Roth, Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen, *Commentarii Mathematici Helvetici* **11** (1938) 77–125.