Exercise on the order of an entire function

An entire function has a Maclaurin series expansion $\sum_{n=0}^{\infty} c_n z^n$ that converges in the entire complex plane. Since the series coefficients c_n uniquely determine the function, the order λ should be computable directly from the c_n. This exercise provides a formula for computing the order of an entire function from the Maclaurin series coefficients.

Recall that when f is an entire function, the quantity $M(r)$ denotes $\max\{\,|f(z)| : |z| = r\,\}$, and λ equals $\limsup_{r\to\infty}\{\log\log M(r)\}/\{\log r\}$. In this exercise, you will prove that the order λ also equals

$$\limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|c_n|}} \qquad \text{(interpret the fraction as 0 if } c_n = 0\text{).}$$

For working purposes, denote the preceding quantity temporarily by β; you need to show that (a) $\beta \le \lambda$ and (b) $\lambda \le \beta$.

1. Fix a positive ϵ. By the definition of order, $M(r) < e^{r^{\lambda+\epsilon}}$ for sufficiently large r. Bound $|c_n|$ for large n by applying Cauchy's estimate with $r = n^{1/(\lambda+\epsilon)}$, and deduce that $\beta \le \lambda + \epsilon$. Let $\epsilon \downarrow 0$.

2. Fix a positive ϵ. Then $|c_n| < n^{-n/(\beta+\epsilon)}$ for sufficiently large n, by the definition of β. Observe that $M(r) \le \sum_{n=0}^{\infty} |c_n|\, r^n$. By splitting the sum where $n \approx (2r)^{\beta+\epsilon}$, show that $\lambda < \beta + 2\epsilon$. Let $\epsilon \downarrow 0$.

The proof of Hadamard's factorization theorem depends on the theory of infinite *products*. Your work above shows that one can also use infinite *series* to construct entire functions with prescribed growth.

3. Let s be an arbitrary positive real number, and suppose that

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/s}}.$$

Show that f is an entire function of order s.