Exercise on the modular group

The *modular group* is the group of linear fractional transformations

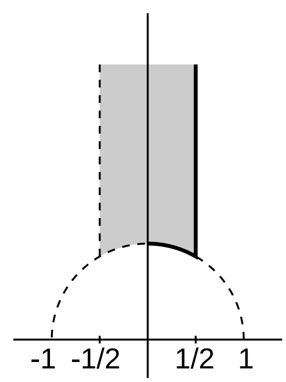
$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1.$$

This group may be viewed as a group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with the understanding that a matrix and its negative are identified (since $\frac{az+b}{cz+d} = \frac{-az-b}{-cz-d}$). The group is known as $PSL(2,\mathbb{Z})$, where $\mathbb{Z}$ indicates integer entries, the number 2 indicates 2×2 matrices, the letter L stands for "linear", the letter S stands for "special" (meaning determinant equal to 1), and the letter P stands for "projective" (referring to the identification of a matrix and its negative).

In the textbook you read about the *congruence subgroup*, which consists of matrices that are congruent modulo 2 to the identity matrix. This exercise addresses instead the full modular group.

1. Show that the translation $z \mapsto z+1$ (corresponding to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) and the mapping $z \mapsto z/(z+1)$ (corresponding to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$) generate the modular group. (In other words, the smallest group containing these two transformations is equal to the modular group.)

The elements of the modular group map the upper half-plane to itself. When z is an arbitrary point in the upper half-plane, the *orbit* of z is the set of all points $\varphi(z)$ obtained as φ runs over the elements of the modular group. A *fundamental domain* for the modular group is a set consisting of one point from each orbit.[1]



2. Show that the shaded region indicated in the figure above is a fundamental domain for the modular group. The region is the union of

$$\{\, z : \operatorname{Im} z > 0 \text{ and } |z| > 1 \text{ and } -\tfrac{1}{2} < \operatorname{Re} z \le \tfrac{1}{2} \,\}$$

and the circular arc $\{\, e^{i\theta} : \pi/3 \le \theta \le \pi/2 \,\}$.

[1] The meaning of the word "domain" here differs from our standard usage, for a fundamental domain is not necessarily either connected or open.