

Exercise on the Weierstrass \wp function

You read in the textbook about a doubly periodic meromorphic function with periods 1 and i and with a double pole at 0. More generally, one can construct an analogous function whose periods are two arbitrary complex numbers with non-real ratio. This family of functions collectively goes by the name of “*the* Weierstrass \wp function”, even though there are infinitely many functions (one for each choice of periods).

This exercise considers, as in the textbook, the case of periods 1 and i (but the generalization to two arbitrary periods would be easy).

1. The *order* of a doubly periodic meromorphic function f with periods 1 and i means the number of poles of f (counted with multiplicity) in any square with sides of length 1 parallel to the coordinate axes such that no poles are on the boundary of the square. For example, the Weierstrass \wp function has order 2. (This usage of the word “order” is unrelated to the concept of the order of an entire function.)

Check that the notion of order is well defined. Then show that for every square with sides of length 1 parallel to the coordinate axes such that $f(z) \neq 0$ for z on the boundary of the square, the number of zeroes of f inside the square (counted with multiplicity) is equal to the order of f .

Hint: recall the argument principle for counting zeroes and poles.

2. Show that \wp' , the derivative of the Weierstrass \wp function, has order 3. Moreover, \wp' has zeroes at the points $1/2$, $i/2$, and $(1+i)/2$, each of these being a simple zero.

Hint: the function \wp' is both an odd function and doubly periodic.

3. Prove the so-called *duplication formula*, which is the identity

$$\wp(2z) = \frac{1}{4} \left(\frac{\wp''(z)}{\wp'(z)} \right)^2 - 2\wp(z).$$

Hint: show that the two sides of the formula have the same poles and the same principal parts at the poles.