

In this assignment, you will enhance your understanding of subharmonic functions and their role in the solution of the Dirichlet problem by Perron's method.

Subharmonicity is a local property that can be characterized in the following ways. (All of these properties are equivalent, modulo assumptions about smoothness of the function.) Suppose Ω is an open subset of \mathbb{C} , and $u: \Omega \rightarrow [-\infty, \infty)$ is an upper semicontinuous function.

Maximum principle The function u is subharmonic on Ω if for every disk in Ω and every harmonic function v on the disk, the difference $u - v$ attains no local maximum (unless $u - v$ reduces to a constant function in the disk).

An equivalent statement is that for every connected open subset of Ω and every harmonic function v on the subset, the difference $u - v$ attains no local maximum (unless $u - v$ reduces to a constant function).

Sub-mean-value property The function u is subharmonic on Ω if for every closed disk $\overline{D}(z_0, r)$ in Ω , the value $u(z_0)$ at the center does not exceed the average value on the boundary:

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

(If u is continuous, then the Riemann integral can be used. For the general case of an upper semicontinuous function u , you need the Lebesgue integral.)

An equivalent statement is that for every point z_0 in Ω , there is a positive radius r_0 such that the sub-averaging property holds on each disk $D(z_0, r)$ for which $0 < r < r_0$.

Positive Laplacian When $u(x, y)$ is twice continuously differentiable, the function $u(x, y)$ is subharmonic if $u_{xx} + u_{yy} \geq 0$, where the subscripts denote second derivatives.

(Even when u is not differentiable, subharmonicity is characterized by a positive Laplacian if you interpret the derivatives in the sense of distributions, or generalized functions.)

1. Give an example of a subharmonic function that is continuous on the whole plane \mathbb{C} , harmonic on no open subset, and
 - a) not bounded below
 - b) radial (a function only of $|z|$), non-negative, and not convex
 (two different problems).
2. Prove the following generalization of Liouville's theorem: If a function is subharmonic on the entire plane \mathbb{C} and is bounded above, then the function must be constant.

Suggestion: The function $A + B \log |z|$ is an available harmonic comparison function on every annulus centered at the origin for every choice of the constants A and B .

Remark: The statement does indeed generalize Liouville's theorem, for if f is a holomorphic function, then $|f|$ is subharmonic.

It is interesting that the statement is specific to the plane: when $n \geq 3$, the space \mathbb{R}^n supports nonconstant subharmonic functions that are bounded above.

3. The Dirichlet problem is not solvable on the punctured unit disk. What goes wrong with Perron's method?

The Perron family for the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ with given boundary values is the family of all subharmonic functions on the punctured disk whose \limsup at the boundary does not exceed the prescribed boundary values. The lack of a subharmonic peaking function at the origin means that one loses control of the Perron family there.

- a) If the boundary values are 1 when $|z| = 1$ and 0 when $z = 0$, then what is the point-wise supremum of the corresponding Perron family on the punctured disk? Why?
- b) If the boundary values are 0 when $|z| = 1$ and 1 when $z = 0$, then what is the point-wise supremum of the corresponding Perron family on the punctured disk? Why?