In this assignment, you will construct the Bergman kernel function, which is the archetype of so-called reproducing kernels.

When Ω is a bounded domain in $\mathbb{C}$, the notation $A^2(\Omega)$ denotes the set of analytic functions on Ω that have square-integrable modulus. This set of functions is a normed vector space:

$$\|f\| \qquad \text{is defined to be} \qquad \left(\int_\Omega |f(z)|^2 \, d\text{Area}_z\right)^{1/2}.$$

The hypothesis that Ω is bounded guarantees that there are some nontrivial functions in $A^2(\Omega)$: namely, all polynomials in z belong to this space. The Cauchy–Schwarz inequality for integrals implies that this normed space additionally is a complex inner-product space:

$$\langle f, g\rangle \qquad \text{is defined to be} \qquad \int_\Omega f(z)\overline{g(z)}\, d\text{Area}_z.$$

It will follow from the discussion below that this inner product space is complete, so $A^2(\Omega)$ is a Hilbert space; in fact, $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

1. a) Use the mean-value property of analytic functions to show that if f is analytic in a neighborhood of the closed disk $\overline{D}(w,r)$, then f satisfies the area-mean-value property: namely, $f(w) = \dfrac{1}{\pi r^2}\displaystyle\int_{D(w,r)} f(z)\, d\text{Area}_z$.

   b) Use the Cauchy–Schwarz inequality for integrals to deduce that if w is a point of Ω whose distance from the boundary of Ω exceeds r, then $|f(w)| \le \frac{1}{r\sqrt{\pi}}\,\|f\|$.

2. Deduce that the unit ball of $A^2(\Omega)$ is a normal family of analytic functions.

3. Let w be an arbitrary point of Ω. Show that among the functions f in $A^2(\Omega)$ for which $f(w) = 1$, there is one function, call it f_w, of minimal norm. Why is f_w unique?

4. Show that if g is a function in $A^2(\Omega)$ such that $g(w) = 0$, then $\langle g, f_w\rangle = 0$.
   Hint: for an arbitrary nonzero complex number λ, the function $f_w + \lambda g$ is an unsuccessful candidate for the solution of the extremal problem in the preceding part.

5. Show that if h is an arbitrary function in $A^2(\Omega)$, then

   $$\langle h, f_w\rangle = h(w)\langle 1, f_w\rangle = h(w)\langle f_w, f_w\rangle.$$

   Hint: if $g(z) = h(z) - h(w)$, then the preceding part applies to g.

6. Define the Bergman kernel function $K(w,z)$ to be $\overline{f_w(z)}/\|f_w\|^2$. Show that

   $$\int_\Omega K(w,z)h(z)\, d\text{Area}_z = h(w) \qquad \text{for every } h \text{ in } A^2(\Omega).$$

Remark: An alternate way to construct f_w and hence $K(w,z)$ is to apply the Riesz representation theorem to the point-evaluation functional. If you know the proof of that theorem, then you may recognize some of the steps above.