1. A number is *algebraic* if it is a zero of a polynomial that has rational coefficients. Examples are $\sqrt[3]{2}$ (which is a zero of the polynomial $z^3 - 2$) and $-1 + i$ (which is a zero of the polynomial $z^2 + 2z + 2$). Numbers that are not algebraic are called *transcendental*. Since the algebraic numbers form a countable set, "most" numbers are transcendental, but determining whether a specific number is transcendental can be a daunting task. Two especially famous numbers known to be transcendental are e (proved by Hermite[1]) and π (proved by Lindemann[2]).

   The transcendental number π is nonetheless a zero of an entire function having a Maclaurin series with rational coefficients (the sine function, for instance). Viewing an entire function as a "polynomial of infinite degree," one might ask which numbers are zeroes of entire power series that have rational coefficients. The amusing answer, due to Hurwitz[3], is that *every* number has this property; moreover, the entire function can be taken to have no other zero.[4]

   Your task is to verify this observation of Hurwitz by showing that if c is an arbitrary complex number, then there exists an entire function $f(z)$ such that the Maclaurin series of $(z-c)e^{f(z)}$ has rational coefficients (in the sense that the real part and the imaginary part of each coefficient are rational numbers). Moreover, if the number c happens to be real, then the coefficients of the series can be taken to be real rational numbers.

   Hint: If $c \neq 0$, then $z - c$ can be written *locally* near the origin in the form $e^{g(z)}$, where $g(z)$ has a series expansion with a finite radius of convergence.

2. Although the proof of the Weierstrass theorem (about prescribed zeroes) is multiplicative, while the proof of the Mittag-Leffler theorem (about prescribed poles and principal parts) is additive, the two theorems can be combined into a unified statement, as Mittag-Leffler himself showed.[5]

   Suppose U is an open set in $\mathbb{C}$, and $\{a_n\}_{n=1}^{\infty}$ is a discrete sequence of distinct points in U. (The word "discrete" here means that the sequence has no accumulation point inside U.) Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be two sequences of polynomials. The claim is that there exists a meromorphic function f on U having no zeroes or poles outside of the set $\{a_n\}$ and such that for each n, the difference

$$f(z) - p_n(1/(z-a_n)) - q_n(z-a_n)$$

[1] Charles Hermite, Sur la fonction exponentielle, *Comptes rendus de l'Académie des Sciences* **77** (1873) 18–24, 74–79, 226–233, 285–293.

[2] F. Lindemann, Ueber die Zahl π, *Mathematische Annalen* **20** (1882) 213–225.

[3] A. Hurwitz, Über beständig convergirende Potenzreihen mit rationalen Zahlencoefficienten und vorgeschriebenen Nullstellen, *Acta Mathematica* **14** (1890) 211–215.

[4] There is a theorem in complex analysis generally known as "Hurwitz's theorem," but that theorem concerns a different topic.

[5] G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," *Acta Mathematica* **4** (1884) 1–79.

has (a removable singularity and) a zero at a_n of order at least $1 + \deg q_n$. In other words, it is possible to prescribe not only the principal part of the Laurent series but also finitely many terms having positive exponents.

Your task is to prove this statement by applying the versions that you already know of the Weierstrass theorem and the Mittag-Leffler theorem. Here is a suggestion for how the argument could go.

As a first step, use the Weierstrass theorem to find a holomorphic function g having for each n a zero at a_n of order $1 + \deg q_n$. Then use the Mittag-Leffler theorem to find a meromorphic function h having for each n the same principal part at a_n as the quotient

$$\frac{p_n(1/(z-a_n)) + q_n(z-a_n)}{g(z)}.$$

The first try is to set f_1 equal to the product gh. You are not yet done, for the standard version of the Mittag-Leffler theorem does not control the location of zeroes: the function f_1 just determined might have some extraneous zeroes lying outside the set $\{a_n\}$.

Next apply the Weierstrass theorem to create a meromorphic function f_2 having zeroes and poles of the same orders as f_1 at the points of the sequence $\{a_n\}$ and having no other zeroes or poles. (The Laurent series coefficients of f_2 are not under control, however.) Locally near each point a_n, you can define a holomorphic branch of $\log(f_1/f_2)$, and by the previous step there is a holomorphic function φ on U that agrees to suitably high order at each a_n with $\log(f_1/f_2)$. Set f equal to $f_2 e^{\varphi}$.