Instructions Solve any *four* of the following six problems. Your solutions are due at the beginning of class on Thursday, March 7.

You may consult your notes from class and the textbook (including sections that we have not officially discussed) but not other sources. In particular, you may not ask another person for help solving the problems. You may, of course, ask me for clarification about the statements of the problems. You may cite and use results stated in the textbook or in class.

1. Weierstrass got the idea for his theory of convergence-producing factors for infinite products by studying the concrete example of the gamma function. In a paper written while he was teaching in a high school,[1] Weierstrass introduced the notation Fc (for "factorial") as follows:[2]

$$\operatorname{Fc}(z) = z \prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{z} \left(1 + \frac{z}{n}\right), \tag{1}$$

where the expression $\left(\frac{n}{n+1}\right)^{z}$ is to be interpreted as $\exp\left(z \log \frac{n}{n+1}\right)$ with the principal branch of the logarithm. Page 209 of the textbook states without proof the following infinite-product representation for the reciprocal of the gamma function:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \tag{2}$$

where γ is Euler's constant, that is,

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

Here is your task:

(a) show that the infinite product in (1) converges uniformly on each compact subset of $\mathbb{C}$; and

(b) show that the infinite products in (1) and (2) agree; that is, $\operatorname{Fc}(z) = 1/\Gamma(z)$ for every complex number z.

[1] Über die Theorie der analytischen Facultäten, *Journal für die reine und angewandte Mathematik* [Crelle's Journal] **51** (1856) 1–60; the byline lists Weierstrass as "Oberlehrer am Gymn. zu Braunsberg" [head teacher in the Braunsberg secondary school].

[2] Modulo change of notation: Weierstrass actually used the variable u instead of z and the summation index α instead of n.

2. Suppose U is a bounded, connected, simply connected, nonvoid open subset of $\mathbb{C}$, and let z_0 be a point in U. Let $\mathcal{F}$ denote the family of holomorphic functions on U such that each f in the family has the following properties:

- the image $f(U)$ is a subset of $\{ w \in \mathbb{C} : |w| < 1 \}$ (the unit disk);
- $f(z_0) = 0$;
- $f'(z_0)$ is a positive real number.

Show that

(a) there is an extremal function in $\mathcal{F}$ for which the value $f'(z_0)$ is maximal; and

(b) this extremal function is equal to the Riemann mapping function.

Remark The set-up is reminiscent of the proof of the Riemann mapping theorem given in the textbook. That proof implicitly shows that if the family $\mathcal{F}$ is further restricted to contain only functions that additionally are injective (univalent), then the extremal function is a biholomorphism from U to the unit disk.

In this problem, however, no a priori assumption is made that the functions in the family $\mathcal{F}$ are injective. The surprise is that the extremal function nonetheless turns out to be both injective and surjective. The main issue is to figure out what magic principle forces the extremal function to be injective.

Suggestion To get started, you might consider the special case that U itself is the unit disk, and $z_0 = 0$.

3. Suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of positive integers such that the infinite series $\sum_{k=1}^{\infty} 1/n_k$ diverges. Prove that if f is a bounded holomorphic function on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane) having a zero at each n_k, then f must be identically equal to zero.

Suggestion Show that the map sending z to $\dfrac{z-1}{z+1}$ is a biholomorphic map from the right-hand half-plane to the unit disk. You know something from a homework exercise about zeroes of bounded holomorphic functions on the unit disk.

4. According to Montel's theorem, a family of holomorphic functions on a (nonvoid) connected open subset of $\mathbb{C}$ is a normal family (in the sense that every sequence of functions in the family admits a subsequence converging uniformly on compact sets to a holomorphic function) if and only if the family is locally bounded (meaning that for every compact subset K there is a constant C such that $|f(z)| \le C$ for every z in K and every f in the family).

Your task is to prove *Marty's theorem*, which provides a necessary and sufficient condition for a family of holomorphic functions on a (nonvoid) connected open subset of $\mathbb{C}$ to be a normal family in the extended sense that every sequence of functions in the family admits either a subsequence that converges uniformly on compact sets to a holomorphic function or a subsequence that tends uniformly on compact sets to the constant ∞. Marty's criterion is that for every compact subset K there is a constant C such that $\dfrac{|f'(z)|}{1+|f(z)|^2} \le C$ for every z in K and every f in the family.

The expression $\dfrac{|f'(z)|}{1+|f(z)|^2}$, sometimes called the *spherical derivative* of f, has the useful property that this expression is unchanged when f is replaced by the reciprocal function $1/f$. Marty's criterion for normality in the extended sense can be rephrased as local boundedness of the family of spherical derivatives.

Suggestions If the family is normal, but Marty's criterion is violated, then there must exist a compact set and a sequence of functions such that …; pass to a convergent subsequence and deduce a contradiction.

In the converse direction, the key issue in deducing normality from Marty's criterion is to prove some uniform control: you need to know that if $|f(z)|$ is small (respectively large) at a point, then $|f(z)|$ remains small (respectively large) in a neighborhood of the point (the neighborhood being independent of the function). One way to get this control is to compute $\dfrac{\partial}{\partial r} \arctan |f(z_0 + re^{i\theta})|$ by the chain rule to deduce that

$$\bigl|\arctan |f(z_0 + Re^{i\theta})| - \arctan |f(z_0)|\bigr| \le \int_0^R \frac{|f'(z_0 + re^{i\theta})|}{1 + |f(z_0 + re^{i\theta})|^2}\, dr.$$

The mean-value theorem from real calculus now tells you something useful.

5. Does there exist a holomorphic function f on $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) with the property that for every sequence $\{z_n\}_{n=1}^{\infty}$ of points in the unit disk for which $|z_n| \to 1$, the corresponding image sequence $\{f(z_n)\}_{n=1}^{\infty}$ is an unbounded subset of $\mathbb{C}$?

Either show how to construct such a function or prove that no such function can exist.

6. Apply Montel's fundamental normality criterion to prove the "little" theorem of Picard: namely, *the image of a nonconstant entire function cannot omit two distinct complex values.*

Suggestions Recall Montel's criterion: a sufficient condition for a family of holomorphic functions on a connected open set U (which could be all of $\mathbb{C}$) to be a normal family in the extended sense is that each function in the family maps U to a subset of $\mathbb{C} \setminus \{0, 1\}$.

Observe that the specific choice of 0 and 1 as the two omitted values is merely a convenient normalization. If instead $f(z)$ never takes the two values a and b, then the function $\dfrac{f(z) - a}{f(z) - b}$ never takes the values 0 and 1.

To prove Picard's little theorem,[3] suppose that f is an entire function whose image does omit the values 0 and 1, and consider the sequence of entire functions $\{f_n\}_{n=1}^{\infty}$, where $f_n(z) = f(nz)$. After invoking Montel's criterion, can you make use of Liouville's theorem?

[3] Picard's "great" theorem, which is a little harder than the little theorem, says that in every punctured neighborhood of an essential singularity, a holomorphic function assumes every complex value—with one possible exception—infinitely often. This more sophisticated theorem too can be proved by applying Montel's fundamental normality criterion. Picard's great theorem subsumes the little theorem, for an entire function either has an essential singularity at infinity or is a polynomial (and a nonconstant polynomial assumes every complex value). Thus one can strengthen the little theorem to say that a nonpolynomial entire function takes every complex value infinitely often, with one possible exception. The exceptional value might be taken not at all or finitely many times.