Instructions Solve any *four* of the following six problems. Your solutions are due at the beginning of class on Thursday, March 7.

You may consult your notes from class and the textbook (including sections that we have not officially discussed) but not other sources. In particular, you may not ask another person for help solving the problems. You may, of course, ask me for clarification about the statements of the problems. You may cite and use results stated in the textbook or in class.

1. Weierstrass got the idea for his theory of convergence-producing factors for infinite products by studying the concrete example of the gamma function. In a paper written while he was teaching in a high school,[1] Weierstrass introduced the notation Fc (for "factorial") as follows:[2]

$$\mathrm{Fc}(z) = z \prod_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{z} \left(1 + \frac{z}{n} \right), \tag{1}$$

where the expression $\left(\frac{n}{n+1} \right)^{z}$ is to be interpreted as $\exp\left(z \log \frac{n}{n+1} \right)$ with the principal branch of the logarithm. Page 209 of the textbook states without proof the following infinite-product representation for the reciprocal of the gamma function:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-z/n}, \tag{2}$$

where γ is Euler's constant, that is,

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right).$$

Here is your task:

(a) show that the infinite product in (1) converges uniformly on each compact subset of $\mathbb{C}$; and

[1] Über die Theorie der analytischen Facultäten, *Journal für die reine und angewandte Mathematik* [Crelle's Journal] **51** (1856) 1–60; the byline lists Weierstrass as "Oberlehrer am Gymn. zu Braunsberg" [head teacher in the Braunsberg secondary school].

[2] Modulo change of notation: Weierstrass actually used the variable u instead of z and the summation index α instead of n.

(b) show that the infinite products in (1) and (2) agree; that is, $\operatorname{Fc}(z) = 1/\Gamma(z)$ for every complex number z.

Solution. To check the normal convergence of (1) in the complex plane, it suffices to verify uniform convergence on an arbitrary closed disk. Accordingly, fix an arbitrary integer N larger than 2, and consider the disk where $|z| \le N/2$. The goal is to prove uniform convergence on this disk of the infinite series

$$\sum_{n=N}^{\infty} \left[z \log\left(\frac{n}{n+1}\right) + \log\left(1 + \frac{z}{n}\right) \right],$$

or equivalently, of the series

$$\sum_{n=N}^{\infty} \left\{ \left[\frac{z}{n} - z\log\left(1+\frac{1}{n}\right) \right] - \left[\frac{z}{n} - \log\left(1 + \frac{z}{n}\right) \right] \right\}.$$

Exercise 31.2 in the textbook says that $|\log(1+w) - w| \le |w|^2$ when $|w| \le 1/2$. Since $|z/n| \le 1/2$ when z is in the indicated disk and $n \ge N$, the exercise implies that the modulus of the general term of the infinite series is bounded above by

$$|z| \cdot \frac{1}{n^2} + \left|\frac{z}{n}\right|^2, \qquad \text{hence by} \qquad \frac{2N^2}{n^2}.$$

Thus the infinite series converges absolutely and uniformly on an arbitrary disk by comparison with the convergent series $\sum_n n^{-2}$.

To verify the equivalence of (1) and (2), observe that

$$\begin{aligned}
\prod_{k=1}^{n} \left(\frac{k}{k+1}\right)^{z} \left(1+\frac{z}{k}\right) &= \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) \exp\left(-\frac{z}{k}\right) \exp\left(\frac{z}{k} + z\log\frac{k}{k+1}\right) \\
&= \left\{ \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) \exp\left(-\frac{z}{k}\right) \right\} \exp\left\{ z \sum_{k=1}^{n} \left(\frac{1}{k} + \log\frac{k}{k+1}\right) \right\}.
\end{aligned}$$

Now $\displaystyle\sum_{k=1}^{n} \log\frac{k}{k+1}$ telescopes to $-\log(n+1)$, and

$$\begin{aligned}
&\lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \log(n+1) \right] \\
&\qquad = \lim_{n\to\infty} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) + \log\frac{n}{n+1} \right] = \gamma + 0.
\end{aligned}$$

Accordingly,

$$\prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{z} \left(1+\frac{z}{n}\right) = e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n},$$

as claimed.

2. Suppose U is a bounded, connected, simply connected, nonvoid open subset of $\mathbb{C}$, and let z_0 be a point in U. Let $\mathcal{F}$ denote the family of holomorphic functions on U such that each f in the family has the following properties:

- the image $f(U)$ is a subset of $\{\, w \in \mathbb{C} : |w| < 1 \,\}$ (the unit disk);
- $f(z_0) = 0$;
- $f'(z_0)$ is a positive real number.

Show that

(a) there is an extremal function in $\mathcal{F}$ for which the value $f'(z_0)$ is maximal; and

(b) this extremal function is equal to the Riemann mapping function.

Remark The set-up is reminiscent of the proof of the Riemann mapping theorem given in the textbook. That proof implicitly shows that if the family $\mathcal{F}$ is further restricted to contain only functions that additionally are injective (univalent), then the extremal function is a biholomorphism from U to the unit disk.

In this problem, however, no a priori assumption is made that the functions in the family $\mathcal{F}$ are injective. The surprise is that the extremal function nonetheless turns out to be both injective and surjective. The main issue is to figure out what magic principle forces the extremal function to be injective.

Suggestion To get started, you might consider the special case that U itself is the unit disk, and $z_0 = 0$.

Solution. The family $\mathcal{F}$ contains, in particular, the Riemann mapping function, so $\mathcal{F}$ is not void. Being bounded, the family $\mathcal{F}$ is normal. Take a sequence of functions in $\mathcal{F}$ whose derivatives at z_0 are positive and approach the supremum of all such values. Normality guarantees the existence of a subsequence that converges normally to a holomorphic function, which necessarily still maps z_0 to 0.

Moreover, the corresponding subsequence of derivatives converges normally too. Consequently, the limiting function has a derivative at z_0 that attains the maximal value. The image of the limiting function a priori is a subset of the closed unit disk, but by the open-mapping principle the image actually is a subset of the open unit disk; the limiting function cannot be a constant, for the derivative at z_0 is not zero. In summary, there is an extremal function, and this function lies in the family $\mathcal{F}$.

Let f denote an extremal function (not yet known to be unique), and let R denote the Riemann mapping function that maps U biholomorphically to the unit disk, taking z_0 to 0 and having positive derivative at z_0 (the derivative can be made real and positive by composing with a suitable rotation). The composite function $f \circ R^{-1}$ then maps the unit disk into itself, fixing the origin. Cauchy's estimate for the first derivative implies that $|(f \circ R^{-1})'(0)| \leq 1$. The chain rule implies that $(f \circ R^{-1})'(0) = f'(z_0)/R'(z_0)$, so $f'(z_0) \leq R'(z_0)$. But $f'(z_0) \geq R'(z_0)$ by the extremal property of f, so actually $f'(z_0) = R'(z_0)$. In other words, the composite function $f \circ R^{-1}$ maps the unit disk to itself, fixes the origin, and has derivative equal to 1 at the origin. The desired conclusion that $f = R$ is the content of the following lemma.

Lemma. *Suppose F is a holomorphic function in the unit disk such that $|F(z)| < 1$ when $|z| < 1$ and $F(0) = 0$ and $F'(0) = 1$. Then F is the identity function, $F(z) = z$.*

First proof. Define a function g in the unit disk as follows:

$$g(z) = \begin{cases} F(z)/z, & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

Since $g(0) = 1 = F'(0) = \lim_{z\to 0} F(z)/z$, the function g is holomorphic.

If $0 < r < 1$ and $|z| = r$, then $|g(z)| = |F(z)|/r < 1/r$. Therefore the maximum of $|g|$ on the disk of radius r is at most $1/r$. Taking the limit as $r \to 1$ shows that the supremum of $|g|$ on the unit disk is at most 1. But $g(0) = 1$, so the maximum principle implies that g is a constant function: namely, $g(z)$ is identically equal to 1. Thus $F(z)$ is identically equal to z.

This proof is essentially the proof of the Schwarz lemma, and many authors fold the conclusion into the statement of the Schwarz lemma. □

Second proof. The Maclaurin series of $F(z)$ begins $z + \cdots$. The claim is that the terms in $\cdots$ all are identically equal to 0. Suppose to the contrary that there is a natural number k and a nonzero constant a_k such that $F(z) = z + a_k z^k + \cdots$. Observe that $(F \circ F)(z) = z + 2a_k z^k + \cdots$, and more generally, the n-fold composition of F with itself has series expansion $z + na_k z^k + \cdots$.

The iterates of F all have image contained in the unit disk, so this sequence of functions is normal. Consequently, the sequence of kth derivatives of the iterates is normal, hence locally bounded. But the value at the origin of the kth derivative of the nth iterate equals $k!\, na_k$, which evidently is unbounded when $n \to \infty$. The contradiction shows that the series expansion of $F(z)$ must reduce to z.

This argument works more generally to prove a proposition of Cartan: If a holomorphic mapping of a *bounded* subdomain of $\mathbb{C}^N$ into itself looks like the identity mapping to first order at one point, then the mapping *is* the identity. $\square$

3. Suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of positive integers such that the infinite series $\sum_{k=1}^{\infty} 1/n_k$ diverges. Prove that if f is a bounded holomorphic function on $\{\, z \in \mathbb{C} : \operatorname{Re}(z) > 0 \,\}$ (the right-hand half-plane) having a zero at each n_k, then f must be identically equal to zero.

Suggestion Show that the map sending z to $\dfrac{z-1}{z+1}$ is a biholomorphic map from the right-hand half-plane to the unit disk. You know something from a homework exercise about zeroes of bounded holomorphic functions on the unit disk.

Solution. Evidently the rational function $\dfrac{z-1}{z+1}$ is holomorphic when $z \neq -1$ (and in particular, in the right-hand half-plane). Rewriting the fraction as $1 - \dfrac{2}{z+1}$ shows that the function is injective (since translation, dilation, and inversion are). Since

$$\left|\frac{z-1}{z+1}\right|^2 = 1 - \frac{4\operatorname{Re}(z)}{|z+1|^2},$$

the image of the right-hand half-plane lines inside the unit disk. If $w = \dfrac{z-1}{z+1}$, then $z = \dfrac{1+w}{1-w}$, and

$$\operatorname{Re}\left(\frac{1+w}{1-w}\right) = \frac{1-|w|^2}{|1-w|^2},$$

so the inverse function maps the open unit disk into the right-hand half-plane. In summary, the function sending z to $\dfrac{z-1}{z+1}$ is indeed a biholomorphic mapping from the right-hand half-plane onto the unit disk.

Composing f with the inverse mapping creates a bounded holomorphic function g on the unit disk having a zero at each point $1 - \dfrac{2}{n_k + 1}$. Since the series $\sum_k \dfrac{1}{n_k}$ diverges, so does $\sum_k \dfrac{2}{n_k+1}$. Consequently, what needs to be shown is that if g is a bounded holomorphic function on the unit disk having zeroes at some sequence $\{a_k\}_{k=1}^{\infty}$ of points in the interval $(0, 1)$ of the positive real axis such that $\sum_k (1 - a_k)$ diverges, then g is identically equal to zero.

Seeking a contradiction, suppose that g is not identically equal to 0. Notice that if $g(0) = 0$, then there is some natural number s such that $g(z)/z^s$ has a finite, nonzero limit at the origin. This new function $g(z)/z^s$ still has a zero at each point a_k. Moreover, the function $g(z)/z^s$ is bounded by the same value that bounds the original function g. (Apply the maximum principle on a disk of radius r less than 1 and take the limit as $r \to 1$.) Accordingly, there is no loss of generality in supposing from the start that $g(0) \neq 0$. Indeed, one may as well assume that $g(0) = 1$ (after multiplying g by a suitable constant).

Method 1: Blaschke products For each natural number n, define a function g_n in the unit disk as follows:

$$g_n(z) = \frac{g(z)}{\prod_{k=1}^{n} \dfrac{a_k - z}{1 - a_k z}}.$$

Remove the removable singularities in this expression. Let M be an upper bound for the modulus of g on the unit disk. Because the Blaschke product has modulus equal to 1 on the boundary of the disk, applying the maximum principle on a disk of radius r less than 1 and taking the limit as $r \to 1$ shows that each g_n has modulus bounded above by M on the unit disk. In particular, $|g_n(0)| \leq M$, or

$$\frac{1}{\prod_{k=1}^{n} a_k} \leq M, \qquad \text{or} \qquad \prod_{k=1}^{n} a_k \geq \frac{1}{M}.$$

Since the partial product $\prod_{k=1}^{n} a_k$ decreases monotonically as n increases and also is bounded below by $1/M$, the infinite product $\prod_{k=1}^{\infty} a_k$ converges (and to a value no smaller than $1/M$). Since $a_k = 1 - (1 - a_k)$, the general theory of infinite products implies that $\sum_{k=1}^{\infty} (1 - a_k)$ converges.

This conclusion contradicts the hypothesis. Consequently, the supposition that g is not identically equal to 0 is untenable.

Method 2: Jensen's theorem Some students discovered how to finish the proof by invoking Jensen's theorem from §32B (which we have not discussed in class). The proof of Jensen's theorem in the textbook uses contour integration, but another proof is based on Blaschke products (as indicated in Exercise 32.1), so Method 2 is not essentially different in spirit from Method 1.

If $0 < r < 1$, and g is not identically equal to 0, then g has finitely many zeroes inside the disk of radius r, say $b_1, \ldots, b_N$ (with repetition according to multiplicity). Bump the value r slightly, if necessary, to ensure that no zero of g has modulus exactly equal to r. Jensen's formula from page 210 says that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\,\mathrm{d}\theta - \log|g(0)| = \sum_{k=1}^{N} \log\frac{r}{|b_k|}.$$

Under the normalization that $g(0) = 1$, and with M again being an upper bound for the modulus of g on the unit disk, the formula implies the following:

$$\sum_{k=1}^{N} \log\frac{r}{|b_k|} \le \log M.$$

Each summand on the left-hand side is positive, so when r increases, the additional terms from extra zeroes can be discarded, and the inequality remains in force. Consequently, taking the limit as $r \to 1$ implies that

$$\sum_{k=1}^{N} \log\frac{1}{|b_k|} \le \log M$$

for each finite number of zeroes of g. The sequence $\{a_n\}_{n=1}^{\infty}$ might not be a complete list of the zeroes of g, but each a_n is eventually among the b's, so

$$\sum_{n=1}^{\infty} \log\frac{1}{a_n} \le \log M.$$

Since

$$\lim_{a\to 1^-} \frac{\log(1/a)}{1-a} = 1,$$

the limit comparison test for infinite series implies that $\sum_{n=1}^{\infty}(1-a_n)$ converges.

As in Method 1, this contradiction establishes the required result.

4. According to Montel's theorem, a family of holomorphic functions on a (nonvoid) connected open subset of $\mathbb{C}$ is a normal family (in the sense that every sequence of functions in the family admits a subsequence converging uniformly on compact sets to a holomorphic function) if and only if the family is locally bounded (meaning that for every compact subset K there is a constant C such that $|f(z)| \le C$ for every z in K and every f in the family).

Your task is to prove *Marty's theorem*, which provides a necessary and sufficient condition for a family of holomorphic functions on a (nonvoid) connected open subset of $\mathbb{C}$ to be a normal family in the extended sense that every sequence of functions in the family admits either a subsequence that converges uniformly on compact sets to a holomorphic function or a subsequence that tends uniformly on compact sets to the constant ∞. Marty's criterion is that for every compact subset K there is a constant C such that $\dfrac{|f'(z)|}{1+|f(z)|^2} \le C$ for every z in K and every f in the family.

The expression $\dfrac{|f'(z)|}{1+|f(z)|^2}$, sometimes called the *spherical derivative* of f, has the useful property that this expression is unchanged when f is replaced by the reciprocal function $1/f$. Marty's criterion for normality in the extended sense can be rephrased as local boundedness of the family of spherical derivatives.

Suggestions If the family is normal, but Marty's criterion is violated, then there must exist a compact set and a sequence of functions such that …; pass to a convergent subsequence and deduce a contradiction.

In the converse direction, the key issue in deducing normality from Marty's criterion is to prove some uniform control: you need to know that if $|f(z)|$ is small (respectively large) at a point, then $|f(z)|$ remains small (respectively large) in a neighborhood of the point (the neighborhood being independent of the function). One way to get this control is to compute $\dfrac{\partial}{\partial r} \arctan |f(z_0 + re^{i\theta})|$ by the chain rule to deduce that

$$\Bigl|\arctan |f(z_0 + Re^{i\theta})| - \arctan |f(z_0)|\Bigr| \le \int_0^R \frac{|f'(z_0 + re^{i\theta})|}{1 + |f(z_0 + re^{i\theta})|^2}\, dr.$$

The mean-value theorem from real calculus now tells you something useful.

Solution. Suppose Marty's criterion is violated. Then there is a compact set K, a sequence $\{z_n\}_{n=1}^{\infty}$ in K, and a sequence $\{f_n\}_{n=1}^{\infty}$ in the family such that

$$\frac{|f_n'(z_n)|}{1+|f_n(z_n)|^2} > n.$$

The claim is that the sequence $\{f_n\}_{n=1}^{\infty}$ is not normal in the extended sense, and therefore the original family is not normal in the extended sense.

If the sequence $\{f_n\}_{n=1}^{\infty}$ had a subsequence $\{f_{n_k}\}$ converging uniformly on compact sets to a holomorphic function, then both the sequence $\{f_{n_k}\}$ and the sequence $\{f_{n_k}'\}$ of derivatives would be locally bounded. Thus there would exist a constant M such that $\max_{z\in K} |f_{n_k}'(z)| < M$ for every k. In particular,

$$\frac{|f_{n_k}'(z_{n_k})|}{1+|f_{n_k}(z_{n_k})|^2} \le |f_{n_k}'(z_{n_k})| < M, \qquad \text{with } M \text{ independent of } k,$$

which contradicts the construction of the sequence $\{f_n\}_{n=1}^{\infty}$. On the other hand, if the sequence $\{f_n\}_{n=1}^{\infty}$ had a subsequence $\{f_{n_k}\}$ tending locally uniformly to ∞, then f_{n_k} would be zero-free in a neighborhood of K for large enough k, and $1/f_{n_k}$ would tend to 0 uniformly in a neighborhood of K. Hence the ordinary derivative of the function $1/f_{n_k}$ would tend to 0 uniformly on K, and so would the spherical derivative of $1/f_{n_k}$. But the spherical derivative of of $1/f_{n_k}$ equals the spherical derivative of f_{n_k}, so again the defining property of the sequence $\{f_n\}_{n=1}^{\infty}$ would be contradicted. In other words, if Marty's criterion is violated, then the family cannot be normal in the extended sense.

Conversely, suppose that Marty's criterion does hold. The goal is to show that the family is normal in the extended sense.

Method 1: Extend the Arzelà–Ascoli theorem The notion of equicontinuity makes sense in metric spaces (or even more generally). Inspection of the proof of the Arzelà–Ascoli theorem reveals that a family $\mathcal{F}$ of functions from a planar domain into a metric space (X, d) is normal (has compact closure with respect to the topology of uniform convergence on compact sets) if and only if for each point z in the domain, the family is equicontinuous at z and the set $\{ f(z) : f \in \mathcal{F} \}$ has compact closure in X.

In particular, the metric space (X, d) can be the extended complex numbers provided with the spherical metric. This metric space is compact, so the second hypothesis in the Arzelà–Ascoli theorem holds trivially. Thus a family of holomorphic functions viewed as maps into the extended complex numbers is normal (with respect to the spherical metric) if and only if the family is equicontinuous with respect to the spherical metric.

Being a normal family with respect to the spherical metric is precisely the notion of normality in the extended sense. Thus the problem reduces to showing that Marty's condition implies equicontinuity with respect to the spherical metric.

In Math 617 last semester, we worked out that the spherical distance between complex numbers z_1 and z_2 equals

$$\frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}$$

(up to a scale factor that depends on the particular model for stereographic projection). The infinitesimal interpretation of this expression is that the differential of arclength for the spherical distance is

$$\frac{|\mathrm{d}z|}{1 + |z|^2},$$

where $|\mathrm{d}z|$ denotes the ordinary arclength differential.

Fix a point z_1, a closed disk centered at z_1 and contained in the domain of the functions, and a point z_2 in this disk. Let γ denote the straight-line path joining z_1 to z_2. Consider an arbitrary function f in the family. The image path $f(\gamma)$ joins $f(z_1)$ to $f(z_2)$ but is not necessarily a geodesic (a path of minimal length), so the spherical length of the path $f(\gamma)$ is an upper bound for the spherical distance between $f(z_1)$ and $f(z_2)$. The spherical length of $f(\gamma)$ is

$$\int_{f(\gamma)} \frac{|\mathrm{d}w|}{1 + |w|^2}, \qquad \text{or} \qquad \int_\gamma \frac{|f'(z)|}{1 + |f(z)|^2}\,|\mathrm{d}z|\,.$$

By Marty's criterion, there is a positive constant C (independent of f) that bounds the integrand of the second integral. Consequently, the spherical distance between $f(z_1)$ and $f(z_2)$ does not exceed $\int_\gamma C\,|\mathrm{d}z|$, or $C|z_1 - z_2|$. So if $|z_1 - z_2| < \varepsilon/C$, then the spherical distance between $f(z_1)$ and $f(z_2)$ is less than ε. The constants are independent of f, so the family is equicontinuous at the arbitrary point z_1.

Thus Marty's condition implies equicontinuity with respect to the spherical metric, as claimed.

Method 2: Local stability estimates Normality is a local property. Fix an arbitrary point z_0 in the domain. It suffices to show that there is a neighborhood of z_0

such that for every sequence $\{f_n\}_{n=1}^{\infty}$ of functions in the family, either there is subsequence that is bounded in the neighborhood, or there is a subsequence for which the reciprocal functions are bounded in the neighborhood. For then there is either a subsequence converging uniformly in a slightly smaller neighborhood, or there is a subsequence for which the reciprocals converge uniformly in a slightly smaller neighborhood. In the second case, Hurwitz's theorem implies that the limit function either is nowhere zero in the neighborhood (and then the original subsequence converges to a holomorphic function) or is identically zero in the neighborhood (and then the original subsequence converges to ∞).

If the closed disk of radius R_0 centered at z_0 is contained in the domain of the functions, then Marty's criterion supplies a positive constant C such that

$$\frac{|f'(z)|}{1+|f(z)|^2} \le C \qquad \text{when } |z-z_0| \le R_0 \text{ and } f \text{ is in the family.}$$

Observe that

$$\frac{\partial}{\partial r}\arctan\left|f(z_0+re^{i\theta})\right| = \frac{1}{1+\left|f(z_0+re^{i\theta})\right|^2}\cdot\frac{\operatorname{Re}\left[e^{i\theta}f'(z_0+re^{i\theta})\overline{f(z_0+re^{i\theta})}\right]}{\left|f(z_0+re^{i\theta})\right|},$$

so

$$\left|\frac{\partial}{\partial r}\arctan\left|f(z_0+re^{i\theta})\right|\right| \le \frac{\left|f'(z_0+re^{i\theta})\right|}{1+\left|f(z_0+re^{i\theta})\right|^2} \le C$$

when $r \le R_0$. Choose a positive radius R_1 less than R_0 and small enough that $R_1C < \pi/12$. Integrating the radial derivative shows that if $|z-z_0| \le R_1$, then

$$\left|\arctan|f(z)| - \arctan|f(z_0)|\right| < \pi/12. \tag{3}$$

Two positive real numbers whose arctangents are close need not themselves be close (for instance, both arctan 100 and arctan 500 are close to $\pi/2$), but either the numbers or their reciprocals must be close. Inequality (3) provides a quantitative estimate, as follows.

If $|f(z_0)| \le 1$, then $\arctan|f(z_0)| \le \pi/4$, so inequality (3) implies that $\arctan|f(z)| < \pi/3$, whence $|f(z)| < \sqrt{3}$. On the other hand, if $|f(z_0)| > 1$, then $\arctan|f(z_0)| > \pi/4$, so inequality (3) implies that $\arctan|f(z)| > \pi/6$, whence $|f(z)| > 1/\sqrt{3}$. Thus Marty's criterion guarantees that the values of functions in the family are uniformly stable on small disks.

Now if $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in the family, then either there are infinitely many values of n for which $|f_n(z_0)| \le 1$, or there are infinitely many values of n for which $|f_n(z_0)| > 1$. Pass to the corresponding subsequence. In the first case, the moduli of the functions in the subsequence are uniformly bounded above by $\sqrt{3}$ on the disk centered at z_0 of radius R_1. In the second case, the moduli of the functions in the subsequence are uniformly bounded below by $1/\sqrt{3}$. Thus the required local estimates hold that guarantee normality of the family in the extended sense.

5. Does there exist a holomorphic function f on $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) with the property that for every sequence $\{z_n\}_{n=1}^{\infty}$ of points in the unit disk for which $|z_n| \to 1$, the corresponding image sequence $\{f(z_n)\}_{n=1}^{\infty}$ is an unbounded subset of $\mathbb{C}$?

Either show how to construct such a function or prove that no such function can exist.

Solution. Seeking a contradiction, suppose that such a function f exists.

The first observation is that f has only finitely many zeroes. Indeed, if there were an infinite sequence of zeroes, then f would trivially be bounded on that sequence, so the unboundedness property of f would imply that the sequence of zeroes does not accumulate at the boundary. By the Bolzano–Weierstrass property, the sequence of zeroes would accumulate in the interior, and then f would be identically equal to zero (by the coincidence principle).

Since f has finitely many zeroes, there is a polynomial p such that f/p is holomorphic and zero-free in the disk. The function f/p inherits the property of being unbounded on every sequence that tends to the boundary. Therefore there is no loss of generality in supposing from the start that f has no zeroes.

After this normalization, the function $1/f$ is holomorphic in the disk. Fix a positive number ε. The claim is that there exists a positive radius r less than 1 such that $|1/f(z)| \le \varepsilon$ when $r < |z| < 1$. For if no such r exists, then there would be a sequence $\{z_n\}_{n=1}^{\infty}$ such that $|1/f(z_n)| > \varepsilon$ and $|z_n| \to 1$, which would contradict the unboundedness of the sequence $\{f(z_n)\}$. Knowing that $|1/f(z)| \le \varepsilon$ when $r < |z| < 1$ implies—by the maximum principle—that $|1/f(z)| \le \varepsilon$ for every z in the unit disk. Since ε is arbitrary, it follows that $1/f(z) = 0$, which is impossible.

The contradiction shows that no function f of the indicated form can exist.

6. Apply Montel's fundamental normality criterion to prove the "little" theorem of Picard: namely, *the image of a nonconstant entire function cannot omit two distinct complex values.*

Suggestions Recall Montel's criterion: a sufficient condition for a family of holomorphic functions on a connected open set U (which could be all of $\mathbb{C}$) to be a normal family in the extended sense is that each function in the family maps U to a subset of $\mathbb{C} \setminus \{0, 1\}$.

Observe that the specific choice of 0 and 1 as the two omitted values is merely a convenient normalization. If instead $f(z)$ never takes the two values a and b, then the function $\dfrac{f(z) - a}{f(z) - b}$ never takes the values 0 and 1.

To prove Picard's little theorem,[3] suppose that f is an entire function whose image does omit the values 0 and 1, and consider the sequence of entire functions $\{f_n\}_{n=1}^{\infty}$, where $f_n(z) = f(nz)$. After invoking Montel's criterion, can you make use of Liouville's theorem?

Solution. Since the map that sends z to nz is a bijection of $\mathbb{C}$, the range of f_n is identical to the range of f. Accordingly, Montel's theorem implies that the family $\{f_n\}$ is normal in the extended sense. Since $f_n(0) = f(0)$ for every n, the family is bounded at the origin, so there cannot be a subsequence tending to ∞. Therefore the family is normal in the ordinary sense. In particular, the family is bounded on the closed unit disk. But the range of the restriction of f_n to the unit disk is identical to the range of the restriction of f to the disk of radius n. Thus the range of the restriction of f to the disk of radius n is bounded with a bound independent of n; that is, the function f is bounded on $\mathbb{C}$. By Liouville's theorem, the function f is constant. Thus every entire function whose range omits two values reduces to a constant function.

[3] Picard's "great" theorem, which is a little harder than the little theorem, says that in every punctured neighborhood of an essential singularity, a holomorphic function assumes every complex value—with one possible exception—infinitely often. This more sophisticated theorem too can be proved by applying Montel's fundamental normality criterion. Picard's great theorem subsumes the little theorem, for an entire function either has an essential singularity at infinity or is a polynomial (and a nonconstant polynomial assumes every complex value). Thus one can strengthen the little theorem to say that a nonpolynomial entire function takes every complex value infinitely often, with one possible exception. The exceptional value might be taken not at all or finitely many times.