Exercise on combining the Weierstrass and Mittag-Leffler theorems

Introduction

Although the proof of the Weierstrass theorem (about prescribed zeroes) is multiplicative, and the proof of the Mittag-Leffler theorem (about prescribed singular parts) is additive, the two theorems can be combined into a unified statement, as Mittag-Leffler himself showed.[1]

The Weierstrass theorem prescribes the orders of zeroes of a holomorphic function but not the coefficients of the Taylor series. On the other hand, Mittag-Leffler's theorem produces a meromorphic function with poles not only of prescribed orders but also with prescribed principal parts. A natural question is whether a holomorphic function can be constructed with finitely many coefficients of the Taylor series prescribed at given points.

The goal of this exercise is to provide an affirmative answer both to this question and to a more general question for meromorphic functions. You will show how to prescribe at given points finitely many terms of the Laurent series, including some terms involving positive powers of the variable. In particular, you will generalize Exercises 1 and 5 in §3 of Chapter VIII of the textbook.

Throughout, suppose that G is an open subset of $\mathbb{C}$ (and G may be assumed connected without loss of generality). Let E be a specified discrete subset of G (a subset having no accumulation point inside G). The set E is necessarily countable, so you can view E as a sequence if you like. (Possibly E is a finite set, but the case of E being infinite is the situation of most interest.)

Interpolation

Your first task is to solve an interpolation problem. Suppose given a complex number c_b for each point b in E. Prove the existence of a holomorphic function f on G with the property that $f(b) = c_b$ for each b in E. In other words, there exists a holomorphic function with prescribed values at the points of a discrete set.

Suggestion The Weierstrass theorem provides a holomorphic function g with a simple zero at b for each b in E. The basic version of Mittag-Leffler's theorem provides a meromorphic function h with principal part $a_b/(z-b)$ at b for each b, the value of the complex number a_b being at your disposal. Adjust a_b to guarantee that the product gh solves the interpolation problem.

Prescribing finitely many coefficients

An initial chunk of the Laurent series at b of a meromorphic function can be viewed as the sum of a polynomial in $1/(z-b)$ and a polynomial in $(z-b)$. Suppose given two families of polynomials,

[1] G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," *Acta Mathematica* **4** (1884) 1–79.

$\{ p_b : b \in E \}$ and $\{ q_b : b \in E \}$. Your next task is prove the existence on G of a meromorphic function f (whose poles lie in E) such that for each b in E, the difference

$$f(z) - p_b(1/(z-b)) - q_b(z-b)$$

has a removable singularity at b and a zero at b of order at least $1 + \deg q_b$. In other words, you can prescribe not only the principal part of the Laurent series (the terms having negative exponents) but additionally a finite number of terms having positive exponents.

Remark The special case that each polynomial p_b is identically zero produces a holomorphic function with finitely many terms of the Taylor series prescribed at each point b, thus generalizing the interpolation problem.

Suggestion Use the Weierstrass theorem to find a holomorphic function g having for each b a zero at b of order $1 + \deg q_b$. Then use the basic version of Mittag-Leffler's theorem to find a meromorphic function h having for each b the same principal part at b as the quotient

$$\frac{p_b(1/(z-b)) + q_b(z-b)}{g(z)}.$$

Set f equal to the product gh and verify that the required properties hold.

Removing extraneous zeroes

So far you have constructed a meromorphic function with finitely many terms of the Laurent series prescribed at points of E, and the function has no poles outside of E. The constructed function might, however, have some zeroes at points outside of E, for the basic version of Mittag-Leffler's theorem does not control the location of zeroes. Your final task is to modify the construction in such a way that the prescribed terms of the Laurent series at the given points are left unchanged, but the final meromorphic function has neither zeroes nor poles outside of E.

Suggestion Let f denote the meromorphic function already constructed. Apply the theorem of Weierstrass to construct a meromorphic function φ having zeroes and poles of the same orders as f at the points of E and having no other zeroes or poles. (But the Weierstrass theorem gives no control on the coefficients of the Laurent series of φ.) Locally near each point b, the quotient f/φ has a removable singularity and is nonzero (why?), so there exists a local holomorphic branch of $\log(f/\varphi)$. By a previous construction, you know how to build a holomorphic function ψ on G that agrees to suitably high order (how high?) at each b with $\log(f/\varphi)$. Verify that φe^{ψ} is the required final meromorphic function.