Midterm Examination

Instructions Solve *four* of the following six problems.

1. State
   a) the Riemann mapping theorem, and
   b) the Weierstrass factorization theorem.

   Solution. See Theorems 4.2 and 5.14 in Chapter VII of the textbook.

2. For which values of the complex variable z does the infinite product $\prod_{n=1}^{\infty}(1-z^n)$ converge? Explain.

   Solution. A necessary condition for an infinite product $\prod_n(1+a_n)$ to converge to a nonzero value is that $\lim_{n\to\infty} a_n = 0$. Consequently, a necessary condition for convergence of the given product to a nonzero value is that $z^n \to 0$ when $n \to \infty$. This property holds if and only if $|z| < 1$.

   Moreover, a sufficient condition for convergence of $\prod_n(1+a_n)$ is convergence of the infinite series $\sum_n |a_n|$. The geometric series $\sum_{n=1}^{\infty} |z^n|$ converges if and only if $|z| < 1$. No term in the product is zero when $|z| < 1$, so the convergent product is nonzero.

   Combining the preceding two observations shows that the given infinite product $\prod_{n=1}^{\infty}(1-z^n)$ converges to a nonzero value if and only if $|z| < 1$. The unit disk is the largest open set on which the product represents an analytic function.

   Remark. More can be said. If $|z| > 1$, then $|1-z^n| \to \infty$ when $n \to \infty$, so the product diverges to infinity (in the sense that the partial products tend to infinity with respect to the spherical metric).

   If z lies on the unit circle and happens to be a root of unity [in other words, there are integers j and k such that $z = \exp(2\pi i j/k)$], then the partial products are eventually equal to zero and hence have limit zero. But one ought not to say that the infinite product converges in this case, for there are infinitely many terms equal to zero. For instance, if $z = -1$, then the product is $2 \cdot 0 \cdot 2 \cdot 0 \cdots$, which essentially has the structure of $\infty \cdot 0$, best regarded as an indeterminate expression.

   If z lies on the unit circle and is not a root of unity [in other words, there is an irrational number r such that $z = \exp(2\pi i r)$], then no term in the product is equal to zero. Since $1 - z^n \not\to 1$, the infinite product cannot converge to a nonzero number. One of you asked whether the infinite product diverges to zero (that is, the partial products have limit equal to zero) or whether the product fails to converge because of oscillation or unboundedness. I did not intend this question about irrational points on the boundary to be part of the examination problem. Indeed, I do not know the answer. I offer this question now as a little research project to any of you who are interested.

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3. Suppose f is an analytic function that maps $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane) into $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk), and $f(1) = 0$. How big can $|f(2)|$ be?

   Solution. According to the proof of the Riemann mapping theorem, the value $|f(2)|$ is maximized among injective analytic functions by the Riemann mapping function, which maps the half-plane biholomorphically onto the disk. This biholomorphism is the linear fractional transformation that sends z to $\dfrac{z-1}{z+1}$. [The Riemann map is not unique, but the only flexibility is post-composing with a rotation, which does not change the absolute value of $f(2)$.] Accordingly, the extremal value equals $\dfrac{2-1}{2+1}$, or $\dfrac{1}{3}$.

   I stated in class that the Riemann mapping function is actually extremal for all analytic functions, whether injective or not. The proof is to consider the composition of an arbitrary analytic function f with the inverse of the Riemann mapping function, call it g. If f is normalized to take the point 1 to 0, then $f \circ g$ takes the unit disk into itself, fixing the origin. The Schwarz lemma implies that $|f \circ g(z)| \le |z|$ for every point z in the disk. In particular,

   $$|f(2)| = |f(g(1/3))| \le 1/3,$$

   so the value of $|f(2)|$ cannot exceed $1/3$.

4. Let $\mathcal{S}$ denote the set of injective analytic functions on the unit disk satisfying the following normalization: if $f \in \mathcal{S}$, then $f(0) = 0$ and $f'(0) = 1$. Results of A. Hurwitz and P. Koebe from over a century ago imply that $\mathcal{S}$ is a normal family.

   Assuming this theorem, deduce that the set $\left\{ \dfrac{1}{f'} : f \in \mathcal{S} \right\}$ of reciprocals of derivatives of functions in $\mathcal{S}$ is a normal family too.

   Solution. Notice that each function in $\mathcal{S}$ is injective, so the derivative is zero-free. Hence the reciprocal of the derivative is analytic (not merely meromorphic).

   Method 1. If the family fails to be locally bounded, then there must be a compact set K, a sequence $\{z_n\}$ of points in K, and a sequence $\{f_n\}$ of functions in $\mathcal{S}$ such that $|1/f_n'(z_n)|$ tends to infinity when $n \to \infty$. In other words, $f_n'(z_n) \to 0$.

   Compactness implies that there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that z_{n_k} tends to a limiting point z^* in K and f_{n_k} tends normally to a limiting analytic function f. Normal convergence is inherited by derivatives, so $f_{n_k}' \to f'$ uniformly on compact sets. In particular, $f_{n_k}'(0) \to f'(0)$, so $f'(0) = 1$; and $f_{n_k}'(z_{n_k}) \to f'(z^*)$, so $f'(z^*) = 0$.

   Thus f' has a zero but is not identically zero. But f' is a normal limit of zero-free analytic functions, so Hurwitz's theorem is contradicted. The contradiction shows that the family of reciprocals of derivatives must be normal after all.

Method 2. Normality is inherited by derivatives, so $\{ f' : f \in \mathcal{S} \}$ is normal in the space $C(D, \mathbb{C})$ of continuous complex-valued functions on the unit disk D. Convergence in the Euclidean metric of $\mathbb{C}$ implies convergence in the spherical metric, so the same family is normal when regarded as a subset of $C(D, \mathbb{C}_\infty)$.

By Marty's theorem, the family $\{ \mu(f') : f \in \mathcal{S} \}$ of spherical derivatives is locally bounded. The spherical derivative is unchanged when a function is replaced by the reciprocal function (this property is inherited from the corresponding invariance of the spherical distance under inversion), so the family $\{ \mu(1/f') : f \in \mathcal{S} \}$ is locally bounded. Therefore the family $\left\{ \dfrac{1}{f'} : f \in \mathcal{S} \right\}$ is normal in $C(D, \mathbb{C}_\infty)$. Since the value $1/f'(0)$ is normalized to be equal to 1, no sequence of functions in this family can tend normally to ∞. Therefore the family is actually normal in $C(D, \mathbb{C})$.

Remark. One of you made the observation that $1/f'$ is the derivative of f^{-1}, the inverse function, and all of the inverse functions map into the unit disk, a bounded set. A bounded family of analytic functions is normal, and normality is inherited by derivatives, so the family $\{1/f'\}$ should be normal.

The difficulty with this argument is that domains need to be considered. Each f^{-1} has its own domain (namely, the image $f(\mathbb{D})$, where $\mathbb{D}$ is the unit disk). To study the family $\{f^{-1}\}$, one has to restrict attention to the common domain of all the inverse functions, which is the intersection $\bigcap_{f \in S} f(\mathbb{D})$. Consequently, this argument cannot prove normality on the whole of $\mathbb{D}$.

Some work is required even to show that $\bigcap_{f \in S} f(\mathbb{D})$ has interior points. A famous result known as *Koebe's one-quarter theorem* says that actually $\bigcap_{f \in S} f(\mathbb{D})$ is equal to $B(0; 1/4)$, the disk centered at the origin of radius $1/4$.

5. Suppose $f(z) = \displaystyle\prod_{n=2}^{\infty} \left(1 - \frac{z}{n!}\right)$. Prove that $\displaystyle\lim_{|z| \to \infty} \frac{|f(z)|}{\exp(|z|)} = 0$.

Solution. Since f is a nonconstant entire function, this function cannot be bounded. The point of the problem is to show that f grows slower than the exponential function.

Evidently $|f(z)| \le \displaystyle\prod_{n=2}^{\infty} \left(1 + \frac{|z|}{n!}\right)$. But $1 + A \le \exp(A)$ when A is an arbitrary nonnegative real number, so

$$|f(z)| \le \prod_{n=2}^{\infty} \exp \frac{|z|}{n!} = \exp \sum_{n=2}^{\infty} \frac{|z|}{n!} = \exp\{|z|(e-2)\}.$$

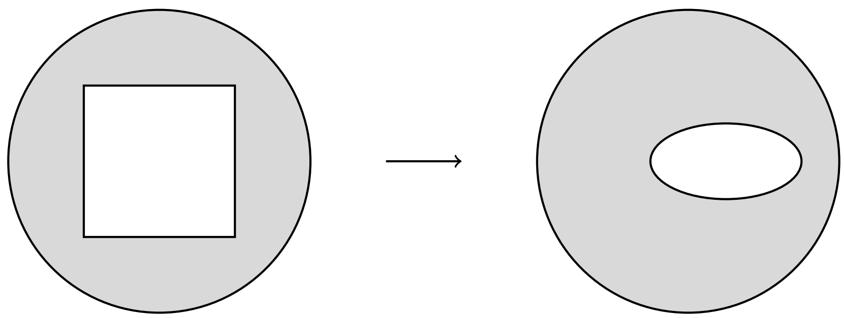
Therefore

$$\frac{|f(z)|}{\exp(|z|)} \le \exp\{|z|(e-3)\}.$$

Since $e - 3 < 0$, the right-hand side tends to zero when $|z| \to \infty$.

Remark. A little more work proves the stronger result that $\lim\limits_{|z|\to\infty} \dfrac{|f(z)|}{\exp(\varepsilon|z|)} = 0$ for every positive ε. We shall explore generalizations of this observation later in the course.

6. Prove that a doubly connected open region bounded on the outside by a circle and on the inside by a square can be mapped biholomorphically to a region bounded by two smooth curves (that is, curves having at each point a well-defined tangent line).
See the figure below.



Solution. First consider the related problem of a region bounded on the *outside* by a square and on the inside by a circle. The Riemann mapping theorem implies that the simply connected region inside the square can be mapped biholomorphically to the unit disk. The mapping carries the circular hole to some hole in the image disk. Why is the image hole bounded by a smooth curve? If the inner circle in the domain is parametrized by a function $\gamma(t)$, then the image of the circle under the conformal mapping f is parametrized by the function $f(\gamma(t))$, the derivative of which is $f'(\gamma(t))\gamma'(t)$. This derivative exists and is nonzero (since the derivative of a conformal map is never zero), so the image curve has a well-defined tangent line.

Notice that this argument does not depend on the outside boundary being a square. The outer boundary could be any curve whatsoever, and the same argument would apply.

To solve the original problem, start with an inversion, which maps the initial region to a region bounded on the inside by a circle and on the outside by a curvilinear quadrilateral. Now apply the preceding argument to the new configuration.

Remark. The same method shows that a finitely connected region whose holes are not single points can be mapped biholomorphically to a region bounded by a finite number of smooth curves.