

Final Examination**Part A**

State *three* of the following theorems.

1. Montel's theorem about locally bounded families of holomorphic functions.
2. Runge's theorem about polynomial approximation.
3. Hadamard's factorization theorem for entire functions.
4. Harnack's principle.
5. Picard's great theorem.

Solution. See the textbook or the lecture notes for the statements of the theorems.

Part B

Pick *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

1. A harmonic function u in the unit disk such that $\lim_{z \rightarrow e^{i\theta}} u(z) = 0$ whenever $0 < \theta < 2\pi$, but $\lim_{z \rightarrow 1} u(z)$ either fails to exist or is different from 0.

Solution. Observe that in the right-hand half-plane, the harmonic function $\operatorname{Re}(z)$ has boundary value equal to 0 on the finite part of the boundary but fails to have a limit at ∞ . Pulling this function back to the unit disk by a suitable linear fractional transformation creates an example for the original problem: namely, $u(z) = \operatorname{Re} \frac{1+z}{1-z}$.

On the other hand, there is no *bounded* example. Indeed, if a harmonic function u is bounded and has limit equal to 0 at every boundary point of the unit disk except possibly the point where $z = 1$, then for an arbitrary positive ε , the harmonic function

$$u(z) + \varepsilon(-\log 3 + \log |1 - z|)$$

has a negative limit at every boundary point (limit $-\infty$ at the boundary point 1). The maximum principle implies that

$$u(z) + \varepsilon(-\log 3 + \log |1 - z|) < 0 \quad \text{for every } z \text{ inside the disk.}$$

Letting ε go to 0 shows that u is a nonpositive function. The same argument applies to $-u$, so u is a nonnegative function too. Therefore u is identically equal to 0.

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2. A holomorphic function f in the unit disk such that the image of f is the twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$.

Solution. The linear fractional map $\frac{z-1}{z+1}$ maps the disk to the left-hand half-plane. Composing with the exponential function maps to the punctured unit disk. Composing with the linear fractional transformation $\frac{1+z}{1-z}$ maps to the right-hand half-plane with a puncture at the point 1. Composing with the principal branch of $z^{5/2}$ maps to the plane with punctures at 0 and 1.

Notice that there cannot be an injective solution, for then there would be an inverse function, but there is no holomorphic surjection from the twice-punctured plane onto the unit disk. (If there were, the two punctures would be removable singularities, and then Liouville's theorem would be contradicted.)

3. An entire function of finite genus with a zero of order $n!$ at each natural number n .

Solution. There is no such function. Finite genus would imply the existence of a natural number k for which the infinite series $\sum_{n=1}^{\infty} \frac{n!}{n^{k+1}}$ converges. The root test (or the ratio test) implies that no such k exists.

4. A continuous function that is subharmonic on the entire plane, constant on some nonvoid open subset, but not identically constant.

Solution. One example is $\max\{1, |z|\}$. The maximum of two subharmonic functions is subharmonic, and this function is constant inside the unit disk but not globally constant.

5. A sequence $\{p_n(z)\}_{n=1}^{\infty}$ of polynomials that is locally bounded in \mathbb{C} , and for each natural number n the degree of p_n equals n , and $\min\{|p_n(z)| : |z| \leq n\}$ exceeds 1 for each n .

Solution. By Montel's theorem, there must be a subsequence of the polynomials that converges normally to an entire function whose absolute value is everywhere at least 1. The reciprocal of this entire function is then bounded, hence constant by Liouville's theorem. Therefore any possible example sequence must be a suitable perturbation of a constant function.

Here is one popular example:

$$p_n(z) = 3 + \left(\frac{z}{n}\right)^n.$$

When $|z| \leq n$, the triangle inequality implies that $|p_n(z)| \geq 2$. Moreover, this sequence of polynomials converges locally uniformly to the constant 3, hence is locally bounded.