Welcome to Math 618

The first big theorem for the semester:

Theorem (Riemann mapping theorem)

A proper subdomain of $\mathbb{C}$ is topologically equivalent to the unit disk if and only if it is holomorphically equivalent to the unit disk.

Remark: The obvious generalization of this theorem to higher dimension is dramatically false.

The metric space $C(K)$

If K is a compact subset of $\mathbb{C}$, then there is a norm on the space $C(K)$ of continuous complex-valued functions on K:

$$\|f\|_K = \max\{\, |f(z)| : z \in K \,\}.$$

A sequence $\{f_n\}_{n=1}^{\infty}$ converges to a limit function f in $C(K)$ when $\|f_n - f\|_K \to 0$, that is, when the sequence of functions converges *uniformly* on K.

Compactness

Theorem (Heine–Borel)

The following properties of a subset K of $\mathbb{C}$ are equivalent:

1. *K is compact,*
2. *K is both closed and bounded.*

Theorem (Arzelà–Ascoli)

When K is a compact subset of $\mathbb{C}$, the following properties of a subset S of $C(K)$ are equivalent:

1. *S is compact,*
2. *S is closed, norm bounded, and uniformly equicontinuous,*
3. *S is closed, pointwise bounded, and pointwise equicontinuous.*

Continuity: theme and variations

Continuity of f at z: For every positive ε there exists a positive δ such that $|f(z) - f(w)| < \varepsilon$ when $|z - w| < \delta$.

Uniform continuity of f on a set: The value of δ can be taken to be independent of z.

Equicontinuity of a family of functions at a point z: The value of δ can be taken to be independent of the function in the family.

Uniform equicontinuity of a family of functions on a set: The value of δ can be taken to be independent of both the function and the point.

Proof of Arzelà–Ascoli, (2) $\implies$ (1)

$C(K)$ is a metric space, so compactness is the same as sequential compactness.

Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ is bounded and uniformly equicontinuous.

The strategy is to find a subsequence that converges at each point of a countable dense subset of K (via a diagonal argument) and then to invoke equicontinuity to show that convergence actually happens uniformly on all of K.

Assignment to hand in next time

Prove that (2) $\iff$ (3) in the Arzelà–Ascoli theorem.