Take-home Midterm Examination

Instructions. Your solutions are due at the beginning of class on Thursday, February 28. You may consult the textbook and the notes from class. If you invoke a theorem or a formula from one of these sources, please state the result that you are using.

1. One of Montel's theorems says that a family $\mathcal{F}$ of analytic functions on an open set G in $\mathbb{C}$ is locally bounded if and only if the family is normal. Here "normal" means that every sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of G to an analytic function (which may or may not belong to the family $\mathcal{F}$). In other words, "normal" means precompact in the metric space $C(G, \mathbb{C})$.

   There is an extended sense of the word "normal" in common use that allows the limit of the convergent subsequence to be either an analytic function or the constant ∞. This extended sense of "normal" amounts to saying that the family $\mathcal{F}$ is precompact in the metric space $C(G, \mathbb{C}_\infty)$.

   Consider the following concrete example. Suppose G is $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) and $\mathcal{F}$ is the family of analytic functions mapping G into $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane). The family $\mathcal{F}$ is not normal in the original sense, for the sequence $\{n\}_{n=1}^{\infty}$ of constant functions has no subsequence that converges to an analytic function. But this sequence does converge to the point ∞ in $\mathbb{C}_\infty$, the extended complex numbers.

   Your task is to prove for this example that the family $\mathcal{F}$ is indeed normal in the extended sense. You need to show that for an arbitrary sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, either there is a subsequence converging uniformly on compact subsets of the unit disk to an analytic function, or there is a subsequence converging uniformly on compact sets to ∞.

   Remark. This problem will become trivial later in the course, after we learn a deep result known as Montel's Fundamental Normality Criterion. The most direct approach using tools that we have available now is probably to compose with the following linear fractional transformation:
   $$\varphi(z) = \frac{1-z}{1+z}.$$
   This function φ is equal to its own inverse (that is, $\varphi \circ \varphi =$ identity), and φ maps the right-hand half-plane bijectively to the unit disk.

2. The Riemann mapping theorem says that if G is a proper connected open subset of $\mathbb{C}$, and if G is simply connected, and if z_0 is a specified base point in G, then there exists a bijective holomorphic function mapping G onto the unit disk and taking z_0 to 0. The standard proof solves an extremal problem in the family of all injective holomorphic functions mapping G into (not necessarily onto) the unit disk and taking z_0 to 0.

   The extremal problem in the textbook maximizes $|f'(z_0)|$ for f in the given family. The proof from class instead chooses a point z_1 in G different from z_0 and maximizes $|f(z_1)|$.

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Choose whichever of these two extremal problems that you prefer, and consider the corresponding extremal problem in the larger family $\mathcal{F}$ of all holomorphic functions (not necessarily injective, not necessarily surjective) mapping G into the unit disk and taking z_0 to 0. Your task is prove that

(a) the extremal problem has a solution within this larger family $\mathcal{F}$, and

(b) this extremal function is a bijective holomorphic function that maps G onto the unit disk.

Hint. Part (b) is easier than appears at first sight. Instead of repeating the whole proof of the Riemann mapping theorem, you can exploit the properties of the extremal function that exists within the standard class of injective holomorphic functions.

3. According to the theorem of Weierstrass, there must exist entire functions having zeros at the integer lattice points in the first quadrant (and at no other points). The goal of this problem is to establish a concrete example of such a function.

   Your task is to prove that

$$\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left(1-\frac{z}{m+in}\right)\exp\left(\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}\right)$$

   converges uniformly on each compact subset of $\mathbb{C}$. Here $\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}$ can be interpreted as $\lim_{M\to\infty}\prod_{m=1}^{M}\lim_{N\to\infty}\prod_{n=1}^{N}$.

4. The function Γ is a meromorphic function that "interpolates" the factorial function at the positive integers, namely, $\Gamma(n+1) = n!$ when $n \in \mathbb{N}$. In 1895, the famous French mathematician Jacques Hadamard (1865–1963) gave an example of an *entire* function F that interpolates the factorial function, namely,

$$F(z) = \frac{\Gamma(z)\sin(\pi z)}{\pi}\,\frac{d}{dz}\log\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)}.$$

   Your task is to verify this example. In other words, prove that

   (a) F is an entire function
   (in the sense that the definition of F does not depend on the choice of the branch of the logarithm, and the apparent singularities in the function are removable), and

   (b) $F(n+1) = n!$ when n is a positive integer.