Exercise on defining the order of an entire function

The notation $M_f(r)$, or $M(r)$ for short, means $\max\{\, |f(z)| \,:\, |z| \le r \,\}$. Liouville's theorem implies that if f is a nonconstant entire function, then $M(r) \to \infty$ as $r \to \infty$. The *order* of f describes how fast $M(r)$ goes to ∞.

Consider the following four numbers associated to a nonconstant entire function $f(z)$ having Maclaurin series $\sum_{n=0}^{\infty} c_n z^n$.

$$\rho := \inf \left\{ t \in (0,\infty) \,:\, |f(z)| \exp(-|z|^t) \text{ is a bounded function of } z \text{ in } \mathbb{C} \right\}$$

$$\sigma := \inf \left\{ t \in (0,\infty) \,:\, \lim_{r\to\infty} \frac{\log M(r)}{r^t} = 0 \right\}$$

$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

$$\beta := \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|c_n|}} \qquad \text{(When } |c_n| = 0\text{, interpret the whole fraction as being 0.)}$$

The main goal is to show that $\rho = \sigma = \lambda = \beta$. In other words, any one of these quantities could be taken as the definition of the order of f. (The choice of the four letters is ad hoc. There is no entirely standard notation for these four quantities.)

1. To check your understanding of the definitions, verify that when $f(z) = ze^z$, each of the four numbers ρ, σ, λ, and β is equal to 1.

2. Verify the existence of an entire function having a prescribed positive order s by showing that if $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/s}}$, then the value of β is equal to s.

Now let f be a general nonconstant entire function, and fix a positive ε.

3. Show that if ρ is finite, then $\log M(r)$ is bounded above by a constant plus $r^{\rho+\varepsilon}$. Deduce that $\sigma \le \rho + 2\varepsilon$.

4. Show that if σ is finite, then $\log M(r) < r^{\sigma+\varepsilon}$ for sufficiently large r. Deduce that $\lambda \le \sigma + \varepsilon$.

5. Show that if λ is finite, then $M(r) < \exp(r^{\lambda+\varepsilon})$ for sufficiently large r. Bound $|c_n|$ for large n by applying Cauchy's estimate with $r = n^{1/(\lambda+\varepsilon)}$. Deduce that $\beta \le \lambda + \varepsilon$.

6. Show that if β is finite, then $|c_n| < n^{-n/(\beta+\varepsilon)}$ for sufficiently large n. Observe that $M(r) \le \sum_{n=0}^{\infty} |c_n|\, r^n$. By splitting the sum where $n \approx (2r)^{\beta+\varepsilon}$, show that $\rho \le \beta + 2\varepsilon$.

Finally, let ε tend to 0 to conclude that $\rho = \sigma = \lambda = \beta$.