Take-home Midterm Examination

Instructions. Your solutions are due at the beginning of class on Thursday, February 28. You may consult the textbook and the notes from class. If you invoke a theorem or a formula from one of these sources, please state the result that you are using.

1. One of Montel's theorems says that a family $\mathcal{F}$ of analytic functions on an open set G in $\mathbb{C}$ is locally bounded if and only if the family is normal. Here "normal" means that every sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of G to an analytic function (which may or may not belong to the family $\mathcal{F}$). In other words, "normal" means precompact in the metric space $C(G, \mathbb{C})$.

   There is an extended sense of the word "normal" in common use that allows the limit of the convergent subsequence to be either an analytic function or the constant ∞. This extended sense of "normal" amounts to saying that the family $\mathcal{F}$ is precompact in the metric space $C(G, \mathbb{C}_\infty)$.

   Consider the following concrete example. Suppose G is $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk) and $\mathcal{F}$ is the family of analytic functions mapping G into $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane). The family $\mathcal{F}$ is not normal in the original sense, for the sequence $\{n\}_{n=1}^{\infty}$ of constant functions has no subsequence that converges to an analytic function. But this sequence does converge to the point ∞ in $\mathbb{C}_\infty$, the extended complex numbers.

   Your task is to prove for this example that the family $\mathcal{F}$ is indeed normal in the extended sense. You need to show that for an arbitrary sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, either there is a subsequence converging uniformly on compact subsets of the unit disk to an analytic function, or there is a subsequence converging uniformly on compact sets to ∞.

   Remark. This problem will become trivial later in the course, after we learn a deep result known as Montel's Fundamental Normality Criterion. The most direct approach using tools that we have available now is probably to compose with the following linear fractional transformation:

$$\varphi(z) = \frac{1 - z}{1 + z}.$$

   This function φ is equal to its own inverse (that is, $\varphi \circ \varphi =$ identity), and φ maps the right-hand half-plane bijectively to the unit disk.

   Solution. Given a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, consider the sequence $\{\varphi \circ f_n\}_{n=1}^{\infty}$ of composite functions. The functions in this new sequence map into the unit disk, so the new sequence is not only locally bounded but even bounded. By Montel's theorem, the new sequence forms a normal family. Consequently, there is an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of natural numbers such that the sequence $\{\varphi \circ f_{n_j}\}_{j=1}^{\infty}$ converges uniformly on compact subsets of the unit disk to some analytic function g.

The range of g must be a subset of the closed unit disk, and if the range of g contains a point of the boundary of the unit disk, then g must be a constant function (by the maximum principle or by the open-mapping theorem). The argument now splits into two cases. Either g is the constant function -1, or the point -1 is not in the range of g.

When $g \equiv -1$, fix a compact subset K of the unit disk and a (large) positive number M. Since φ has a pole at -1, there is a (small) positive ε such that $|\varphi(w)| > M$ when $0 < |w+1| < \varepsilon$. Since $\varphi \circ f_{n_j} \to -1$ uniformly on K, there is a positive integer J such that $|\varphi \circ f_{n_j}(z) + 1| < \varepsilon$ when $z \in K$ and $j > J$. Taking w to be $\varphi \circ f_{n_j}(z)$ reveals that $|\varphi \circ \varphi \circ f_{n_j}(z)| > M$ when $z \in K$ and $j > J$. But $\varphi \circ \varphi$ is the identity map, so $|f_{n_j}(z)| > M$ when $z \in K$ and $j > J$. Since K and M are arbitrary, the subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ converges uniformly on compact sets to ∞.

Now suppose that the point -1 is not in the range of g. Fix a compact subset K of the unit disk. The image set $g(K)$ is compact and does not contain the point -1. Let δ be the distance between the set $g(K)$ and the point -1, and let L be the compact set consisting of all points whose distance from $g(K)$ is less than or equal to $\delta/2$. Since $\varphi \circ f_{n_j} \to g$ uniformly on K, there is a (large) positive integer J such that the range of $\varphi \circ f_{n_j}$ is contained in L when $j > J$. But φ is uniformly continuous on L, so $\varphi \circ \varphi \circ f_{n_j} \to \varphi \circ g$ uniformly on K. In other words, $f_{n_j} \to \varphi \circ g$ uniformly on compact subsets of the unit disk.

Accordingly, the family $\mathcal{F}$ is normal in the extended sense. Every sequence of functions in the family has either a subsequence converging normally to ∞ or a subsequence converging normally to an analytic function.

2. The Riemann mapping theorem says that if G is a proper connected open subset of $\mathbb{C}$, and if G is simply connected, and if z_0 is a specified base point in G, then there exists a bijective holomorphic function mapping G onto the unit disk and taking z_0 to 0. The standard proof solves an extremal problem in the family of all injective holomorphic functions mapping G into (not necessarily onto) the unit disk and taking z_0 to 0.

   The extremal problem in the textbook maximizes $|f'(z_0)|$ for f in the given family. The proof from class instead chooses a point z_1 in G different from z_0 and maximizes $|f(z_1)|$.

   Choose whichever of these two extremal problems that you prefer, and consider the corresponding extremal problem in the larger family $\mathcal{F}$ of all holomorphic functions (not necessarily injective, not necessarily surjective) mapping G into the unit disk and taking z_0 to 0. Your task is prove that

   (a) the extremal problem has a solution within this larger family $\mathcal{F}$, and

   (b) this extremal function is a bijective holomorphic function that maps G onto the unit disk.

Hint. Part (b) is easier than appears at first sight. Instead of repeating the whole proof of the Riemann mapping theorem, you can exploit the properties of the extremal function that exists within the standard class of injective holomorphic functions.

Solution. For part (a), observe that the family $\mathcal{F}$ is bounded, hence—by Montel's theorem—normal (in the original sense of "normal"). And the family $\mathcal{F}$ contains the Riemann mapping function, so is nonvoid.

Let M be $\sup\{\, |f'(z_0)| : f \in \mathcal{F}\,\}$ for the book's extremal problem or $\sup\{\, |f(z_1)| : f \in \mathcal{F}\,\}$ for the extremal problem from class. Take a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $|f_n'(z_0)| \to M$, respectively $|f_n(z_1)| \to M$. By normality, there is a subsequence that converges uniformly on compact subsets of G to some analytic function g. Evidently $g(z_0) = 0$, since the singleton $\{z_0\}$ is a compact set. For the extremal problem from the book, $|g'(z_0)| = M$ (since normal convergence is inherited by derivatives), and for the extremal problem from class, $|g(z_1)| = M$; so g is not a constant function. By the open-mapping theorem, the range of g is an open set. A priori, the range of g is a subset of the closed unit disk, but being open, the range is actually a subset of the open unit disk. Thus the limit function g belongs to the family $\mathcal{F}$.

For part (b), let h denote the extremal function that is known to exist for the family of injective holomorphic functions. In other words, let h be the known (bijective) Riemann mapping function. This function h is a candidate for the solution of the extremal problem in the larger family $\mathcal{F}$, so $|h'(z_0)| \le |g'(z_0)|$ (for the book's extremal problem) or $|h(z_1)| \le |g(z_1)|$ (for the extremal problem from class). The composite function $g \circ h^{-1}$ maps the unit disk into itself, fixing the origin. By the Schwarz lemma, $|(g \circ h^{-1})'(0)| \le 1$ and $|g \circ h^{-1}(w)| \le |w|$ for every point w in the unit disk.

For the extremal problem from the book, observe that $(g \circ h^{-1})'(0) = g'(z_0)/h'(z_0)$, so $|g'(z_0)| \le |h'(z_0)|$. Since inequality holds in the other direction too, the values $|g'(z_0)|$ and $|h'(z_0)|$ are equal. So equality holds in the Schwarz lemma, which implies that $g \circ h^{-1}$ is a rotation. Therefore g is a composition of two bijections, hence a bijection itself.

For the extremal problem from class, apply the deduction that $|g \circ h^{-1}(w)| \le |w|$ with w equal to $h(z_1)$. Then $|g(z_1)| \le |h(z_1)|$. Since inequality holds in the other direction too, the values $|g(z_1)|$ and $|h(z_1)|$ are equal. So equality holds in the Schwarz lemma at a nonzero point, which implies that $g \circ h^{-1}$ is a rotation. Therefore g is a composition of two bijections, hence a bijection itself.

3. According to the theorem of Weierstrass, there must exist entire functions having zeros at the integer lattice points in the first quadrant (and at no other points). The goal of this problem is to establish a concrete example of such a function.

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Your task is to prove that

$$\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left(1-\frac{z}{m+in}\right)\exp\left(\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}\right)$$

converges uniformly on each compact subset of $\mathbb{C}$. Here $\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}$ can be interpreted as $\lim_{M\to\infty}\prod_{m=1}^{M}\lim_{N\to\infty}\prod_{n=1}^{N}$.

Solution. It suffices to show that, for every positive radius R, both of the products converge uniformly on the closed disk $\{\, z \in \mathbb{C} : |z| \le R \,\}$. A proposition from class (part of the proof of the Weierstrass factorization theorem) implies that if $|w| \le 1/2$, then

$$\left|w+\frac{1}{2}w^2+\log(1-w)\right| \le |w|^3.$$

Accordingly, when $|z| \le R$, and $n \ge 2R$, and m is an arbitrary positive integer, the following inequality holds:

$$\left|\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}+\log\left(1-\frac{z}{m+in}\right)\right| \le \frac{R^3}{|m+in|^3}.$$

The main technical point in the solution is the following statement, the proof of which is deferred to the end.

Lemma. *The double series* $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{|m+in|^3}$ *converges.*

A consequence is that the series

$$\sum_{n\ge 2R}\left|\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}+\log\left(1-\frac{z}{m+in}\right)\right|$$

converges uniformly for z in the closed disk of radius R. Since the exponential function is uniformly continuous on compact sets, the product

$$\prod_{n\ge 2R}\left(1-\frac{z}{m+in}\right)\exp\left(\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}\right)$$

converges uniformly on the closed disk of radius R for each fixed positive integer m. Multiplying by a finite number of entire functions preserves the uniform convergence, so the corresponding product $\prod_{n=1}^{\infty}$ converges uniformly on the same disk. Since R is arbitrary, the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{m+in}\right)\exp\left(\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}\right)$$

converges normally on $\mathbb{C}$ for each m.

Similarly, the Lemma implies that the series

$$\sum_{m\geq 2R}\sum_{n=1}^{\infty}\left|\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}+\log\left(1-\frac{z}{m+in}\right)\right|$$

converges uniformly when $|z|\leq R$. Exponentiating shows that the product

$$\prod_{m\geq 2R}\prod_{n=1}^{\infty}\left(1-\frac{z}{m+in}\right)\exp\left(\frac{z}{m+in}+\frac{z^2}{2(m+in)^2}\right)$$

converges uniformly when $|z|\leq R$. Multiplying by the missing finite number of factors preserves the uniform convergence. And since R is arbitrary, the normal convergence of the original double product is demonstrated.

Proof of the lemma. Since all the terms are positive, the order of summation does not matter: the partial sums are monotonically increasing, so the series converges to the least upper bound of the partial sums. All that needs to be shown is that there is an upper bound for the partial sums. There are several ways to prove such an upper bound.

Method 1. Regroup the sum as $\displaystyle\sum_{k=2}^{\infty}\sum_{\substack{m\geq 1\\ n\geq 1\\ m+n=k}}\frac{1}{|m+in|^3}$. If m and n are positive integers with sum k, then one of m and n must be at least $k/2$. Accordingly, $|m+in|>k/2$, and $1/|m+in|^3<8/k^3$. There are $k-1$ pairs of positive integers m and n for which $m+n=k$, so the inner sum is less than $8/k^2$. From first-year calculus, it is known that $\displaystyle\sum_{k=2}^{\infty}8/k^2$ is finite.

Method 2. Since

$$\frac{1}{|m+in|^3}=\frac{1}{|m+in|^{3/2}}\cdot\frac{1}{|m+in|^{3/2}}<\frac{1}{m^{3/2}}\cdot\frac{1}{n^{3/2}},$$

the double sum in the Lemma is bounded above by

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\right)^2,$$

which is known from first-year calculus to be finite.

Method 3. Comparing with the area under a curve reveals that when m is a positive integer,

$$\sum_{n=1}^{\infty} \frac{1}{|m+in|^3} = \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{3/2}} < \int_0^{\infty} \frac{1}{(m^2+x^2)^{3/2}}\,dx.$$

A substitution ($x = mt$) converts the integral to

$$\frac{1}{m^2}\int_0^{\infty} \frac{1}{(1+t^2)^{3/2}}\,dt.$$

This integral with respect to t actually has value equal to 1, but all that is needed is the knowledge that the integral converges. Hence the double sum in the statement of the Lemma is bounded above by a constant times the series $\sum_{m=1}^{\infty} 1/m^2$, which is known to be finite. □

4. The function Γ is a meromorphic function that "interpolates" the factorial function at the positive integers, namely, $\Gamma(n+1) = n!$ when $n \in \mathbb{N}$. In 1895, the famous French mathematician Jacques Hadamard (1865–1963) gave an example of an *entire* function F that interpolates the factorial function, namely,

$$F(z) = \frac{\Gamma(z)\sin(\pi z)}{\pi}\,\frac{d}{dz}\log\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)}.$$

Your task is to verify this example. In other words, prove that

(a) F is an entire function
(in the sense that the definition of F does not depend on the choice of the branch of the logarithm, and the apparent singularities in the function are removable), and

(b) $F(n+1) = n!$ when n is a positive integer.

Solution. The apparent singularities occur at the poles of $\Gamma(z)$ (which are the negative integers and 0), at the poles of $\Gamma(\frac{1}{2}-\frac{z}{2})$ (which are the odd positive integers), and at the poles of $\Gamma(1-\frac{z}{2})$ (which are the even positive integers). What needs to be shown, then, is that F represents an analytic function on $\mathbb{C}\setminus\mathbb{Z}$ and that a finite limit exists at each integer.

On any disk that avoids all the integers, the quotient of Gamma functions is an analytic function without zeros, so there is an analytic logarithm. Since different branches of the logarithm differ locally by a constant, taking the derivative eliminates the ambiguity in the logarithm. Thus F does represent a well-defined analytic function on $\mathbb{C}\setminus\mathbb{Z}$.

At the negative integers and 0, the simple poles of $\Gamma(z)$ are canceled by the corresponding zeros of $\sin(\pi z)$. So the singular points that need attention are the positive integers.

Recall from section V.3 (about the argument principle) that if a function has a simple zero at a point, then the logarithmic derivative has a simple pole at the point and residue 1; and if a function has a simple pole at a point, then the logarithmic derivative has a simple pole at the point and residue -1. Accordingly, the logarithmic derivative of the quotient of Gamma functions has a simple pole at each positive integer n and residue $(-1)^n$. These simple poles are canceled by the remaining zeros of $\sin(\pi z)$, so all the singularities of F are removable.

The knowledge of the residues at these simple poles of the logarithmic derivative reveals that when n is a positive integer,

$$\lim_{z\to n} F(z) = \lim_{z\to n} \frac{\Gamma(z)\sin(\pi z)}{\pi}\left(\frac{(-1)^n}{z-n} + \text{analytic}\right) = \Gamma(n)\cos(\pi n)(-1)^n = \Gamma(n).$$

Thus F and Γ have the same values at the positive integers, as required.

Remark 1. You could alternatively study the behavior of F at the positive integers by bringing in the definition of the Gamma function as an infinite product:

$$\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)} = \frac{\exp\left(-\gamma\left(\frac{1}{2}-\frac{z}{2}\right)\right)}{\exp\left(-\gamma\left(1-\frac{z}{2}\right)\right)}\cdot\frac{1-\frac{z}{2}}{\frac{1}{2}-\frac{z}{2}}\cdot\frac{\prod_{n=1}^{\infty}\left(1+\frac{\frac{1}{2}-\frac{z}{2}}{n}\right)^{-1}\exp\left(\frac{\frac{1}{2}-\frac{z}{2}}{n}\right)}{\prod_{n=1}^{\infty}\left(1+\frac{1-\frac{z}{2}}{n}\right)^{-1}\exp\left(\frac{1-\frac{z}{2}}{n}\right)}.$$

The logarithmic derivative of this expression equals

$$\frac{1}{z-2}-\frac{1}{z-1}+\sum_{n=1}^{\infty}\left(\frac{-1}{z-(2n+1)}-\frac{1}{2n}\right)-\sum_{n=1}^{\infty}\left(\frac{-1}{z-(2n+2)}-\frac{1}{2n}\right),$$

which simplifies to

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-2n} - \frac{1}{z-(2n-1)} \right).$$

Notice that the summands need to be grouped as shown to ensure convergence of the series. The upshot is that when z is not an integer,

$$F(z) = \frac{\Gamma(z)\sin(\pi z)}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{z-2n} - \frac{1}{z-(2n-1)} \right).$$

This expression for F again reveals that the singularities of F are removable and that $F(n) = \Gamma(n)$ when n is a positive integer.

Remark 2. The function F does *not* satisfy the same functional equation as Γ.