Summary of Chapter IX in Conway's book

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1. Main Concepts

   (a) Topological spaces

      i. A topological space is a set together with a collection (called a topology) of subsets (called open sets). A finite intersection and an arbitrary union of open sets should be open; the empty set and the whole set are open by definition.

      ii. A topological space is Hausdorff if for every pair of distinct points there is a pair of disjoint open sets each containing one of the points.

      iii. A topology can be determined by specifying a neighborhood system of each point. The intersection of two neighborhoods (possibly of two different points) should contain a neighborhood of each of its points. A topology is defined by declaring a set to be open if and only if it contains a neighborhood of each of its points.

      iv. The notions of connectedness, compactness, and continuity, discussed in Chapter II for metric spaces, carry over to the setting of topological spaces.

   (b) Analytic manifolds

      i. An analytic manifold (or analytic surface, or complex manifold) is a connected Hausdorff topological space covered by coordinate charts (or patches) homeomorphic to open subsets of $\mathbb{C}$ such that the transition functions between patches are analytic.

ii. Some examples are $\mathbb{C}$, the Riemann sphere $\mathbb{C}_\infty$, the torus (which supports infinitely many different analytic structures), any space that is globally homeomorphic to an open subset of $\mathbb{C}$ or to a previously defined analytic manifold, and the graph of an analytic function with nowhere zero derivative.

iii. Analytic manifolds are both locally and globally arcwise (pathwise) connected.

iv. Analytic functions between analytic manifolds are defined by pulling back to coordinate charts. Local properties of analytic functions in $\mathbb{C}$ carry over to analytic functions on manifolds.

(c) Covering spaces

i. A covering space is a connected topological space together with a continuous mapping onto a base space with the property that each point in the base space has a neighborhood whose inverse image consists of components each of which is mapped homeomorphically by the covering map onto this neighborhood.

ii. Some examples are $\mathbb{R} \to S^1$ via $t \to e^{2\pi i t}$, $S^1 \to S^1$ via $z \to z^n$ for any positive integer n, and $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ via $z \to e^z$.

iii. Covering spaces have the path-lifting property. A path in the base space lifts uniquely to a path in the covering space if the preimage of the initial point is specified. Similarly, homotopies also lift.

(d) Analytic continuation

i. A function element is an analytic function together with a connected open set on which it is defined. A germ of an analytic function at a point is an equivalence class of function elements that agree in some neighborhood of the point.

ii. Analytic continuation of a germ or of a function element along a path corresponds to a family (either continuous or discrete) of disks covering the path and analytic functions in the disks agreeing in the overlaps.

(e) The sheaf of germs of analytic functions

i. The sheaf of germs of analytic functions on an open set in $\mathbb{C}$ consists of germs of analytic functions at points of the set paired with their base points.

ii. A connected component of the sheaf corresponds to all germs that can be reached from a given germ by analytic continuation along paths in the base domain.

iii. A connected component of the sheaf of germs of analytic functions on $\mathbb{C}$ is a Riemann surface. This is, in particular, an analytic manifold and a covering space of some open subset of $\mathbb{C}$. It corresponds to a complete analytic function obtained by making all possible continuations along paths of some initial function element.

2. Main Theorems

   (a) Schwarz reflection principle: A function analytic in one half of a symmetric region and continuous onto the symmetry axis extends by reflection to be analytic in the whole region.
   Idea of proof: The main point is to check analyticity on the symmetry axis. This is a local question that can be answered by applying Morera's theorem. In view of the continuity of the function, an integral over a polygon that cuts the symmetry axis can be replaced, with arbitrarily small error, by two integrals over polygons that do not touch the symmetry axis.

   (b) Monodromy theorem

      i. Version in a domain G in $\mathbb{C}$: If a function element admits unrestricted analytic continuation along paths in G, and if two paths in G are given that are fixed-endpoint homotopic, then analytic continuation of the function element along either path leads to the same result at the common final point.

      ii. Covering space version: If two paths in the base space are fixed-endpoint homotopic, then liftings with the same initial point have the same final point and are also fixed-endpoint homotopic.

   Idea of proof: In the first version, the result is clear if the two paths are very close together. In general, one can build a bridge between the two paths so that every two consecutive intermediate paths are close together. The second version follows immediately from the homotopy lifting property of covering spaces (which in turn can be proved by building a bridge).

(c) Liouville's theorem for analytic manifolds: Every analytic function from a compact analytic manifold into $\mathbb{C}$ is a constant.
Idea of proof: The modulus of such a function would attain its maximum (since its domain is compact). Pulling back to a coordinate chart would give an ordinary analytic function whose modulus attains a local maximum. It would therefore be constant by the usual maximum principle. A connectedness argument shows that the original function is globally constant.