Summary of Chapter VII in Conway's book

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1. Main Concepts

   (a) The space $C(G, \Omega)$ of continuous functions on an open set G in $\mathbb{C}$ taking values in a complete metric space (usually either $\mathbb{C}$ or $\mathbb{C}_\infty$)

      i. $C(G, \Omega)$ carries a metric $\rho(f, g) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}$, where $\rho_n(f, g)$ is the supremum of $d(f(z), g(z))$ as z ranges over K_n, and $\{K_n\}_{n=1}^{\infty}$ is an exhaustion of G by compact sets.

      ii. $C(G, \Omega)$ is a *complete* metric space: Cauchy sequences converge.

      iii. $C(G, \Omega)$ carries the *compact-open* topology: the open sets can be described in the following way. Let $\{K_j\}_{j=1}^{n}$ be a finite sequence of compact subsets of G, and let $\{U_j\}_{j=1}^{n}$ be a finite sequence of open subsets of Ω. Then $\{\, f : f(K_j) \subset U_j$ for $j = 1, \ldots, n \,\}$ is a basic open set in $C(G, \Omega)$, and every open set in $C(G, \Omega)$ is a union of basic open sets.

      iv. Convergence of a sequence of functions in $C(G, \Omega)$ means uniform convergence on each compact subset of G.

   (b) Normal families

      i. A set of functions is normal if it has compact closure in $C(G, \Omega)$. This means that every sequence of functions in the set has a subsequence that converges to a function in $C(G, \Omega)$.

      ii. A family of *holomorphic* functions is normal in $C(G, \mathbb{C})$ if and only if it is locally bounded (Montel's theorem).

iii. A family of *meromorphic* functions is normal in $C(G, \mathbb{C}_\infty)$ if and only if the corresponding family of "spherical derivatives" is locally bounded. The "spherical derivative" of f at z is $\mu(f)(z) = \dfrac{2|f'(z)|}{1+|f(z)|^2} = \lim\limits_{w \to z} \dfrac{d(f(w), f(z))}{|w-z|}$.

(c) Convergence of infinite products

i. For $\prod_{n=1}^\infty z_n$ to have a chance to converge to some *non-zero* complex number, it is necessary that z_n never be zero and that $z_n \to 1$. In that case, convergence of the infinite product is equivalent to convergence of the sum $\sum_{n=1}^\infty \log z_n$.

ii. If in addition $\sum_{n=1}^\infty |\log z_n|$ converges, then the infinite product is *absolutely convergent*, and the terms may be rearranged without changing the limit.

2. Main Theorems

(a) Arzelà-Ascoli theorem: a family $\mathcal{F}$ is normal in $C(G, \Omega)$ if and only if it is equicontinuous at each point of G and has the property that $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω for each fixed z in G.
Idea of proof: Cantor diagonalization process and a 3ϵ argument.

(b) Montel's theorem: a family of holomorphic functions is normal in $C(G, \mathbb{C})$ if and only if it is locally bounded.
Idea of proof: Apply Arzelà-Ascoli. Equicontinuity follows from local boundedness by Cauchy's integral formula.

(c) Hurwitz's theorem: If a holomorphic function f is the limit (uniformly on compact subsets of a connected open set) of nowhere zero holomorphic functions, then f is either nowhere zero or everywhere zero.
Idea of proof: Apply Rouché's theorem.

(d) Riemann mapping theorem: Every simply connected domain in $\mathbb{C}$ (except for $\mathbb{C}$ itself) is conformally equivalent to the unit disk.
Idea of proof: Pick a point that will go to the origin. Use a normal families argument to find a one-to-one holomorphic mapping into the unit disk that maximizes the derivative at the distinguished point. The mapping is shown to be onto by contradiction: if it were not, a new mapping with larger derivative could be cooked

up (using the simple connectivity to take a square root of a certain non-zero function).

(e) Weierstrass factorization theorem

i. Every sequence $\{a_n\}_{n=1}^{\infty}$ tending to ∞ can be the zeroes of an entire function. The most general such entire function has the form

$$z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2 + \cdots + \frac{1}{p_n}(\frac{z}{a_n})^{p_n}},$$

where m is the order of the zero at the origin and g is some entire function. The integers p_n must be chosen to make the infinite product converge; the choice $p_n = n$ is universal and always works.
Idea of proof: Taking logarithms reduces the convergence question to convergence of $\displaystyle\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^{p_n+1}$.

ii. In any open set, there is a holomorphic function with prescribed zeroes of prescribed orders as long as the zeroes do not accumulate inside the region.
Idea of proof: We know how to build a convergent infinite product if the zeroes tend to ∞. If they do not, compose each factor with an inversion to get back to the situation we understand.

3. Main Applications

(a) $\displaystyle\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$.

(b) The gamma function

i. $\Gamma(z)$ is meromorphic in the whole plane, with no zeroes, and with simple poles at 0 and the negative integers. Its values on the positive real axis are uniquely determined by the normalization $\Gamma(1) = 1$, the functional equation $x\Gamma(x) = \Gamma(x+1)$, and the convexity of $\log\Gamma(x)$. When n is a positive integer, $n! = \Gamma(n+1)$.

ii. $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \displaystyle\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$, where γ is Euler's constant chosen to make $\Gamma(1) = 1$: namely, $\gamma = \lim_{n\to\infty}[(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - \log n]$.

iii. Gauss's formula: $\Gamma(z) = \lim\limits_{n\to\infty} \dfrac{n!\, n^z}{z(z+1)\dots(z+n)}$ when z is not one of the poles.

iv. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ when $\operatorname{Re} z > 0$.

(c) Riemann's zeta function

i. $\zeta(z)$ is a meromorphic function in whole plane with only one pole (which is simple) at $z = 1$. It has zeroes at the negative even integers. There are infinitely many other zeroes, all located in the critical strip $0 < \operatorname{Re} z < 1$. The still unproved Riemann hypothesis is that all the zeroes in the critical strip lie on the critical line $\operatorname{Re} z = 1/2$.

ii. By Euler's theorem, $\zeta(z) = \displaystyle\prod_{\text{primes } p} \left(\frac{1}{1 - \left(\frac{1}{p}\right)^z}\right)$ when $\operatorname{Re} z > 1$, so the zeta function is connected with number theory.

iii. $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ when $\operatorname{Re} z > 1$.

iv. Riemann's functional equation is

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\pi z/2).$$