Summary of Chapter VIII in Conway's book

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1. Main Concepts

   (a) Polynomial convexity

      i. A compact set K in $\mathbb{C}$ is polynomially convex if every point in its complement can be separated from K by a polynomial, that is, if $z \notin K$, then there exists a polynomial p such that $|p(z)| > \sup\{|p(w)| : w \in K\}$.

      ii. It follows from Runge's theorem that K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.

      iii. The polynomially convex hull $\widehat{K}$ of K is the smallest polynomially convex set containing K. It is obtained by "filling in the holes" of K, that is, by adding to K all bounded components of $\mathbb{C} \setminus K$.

   (b) Simple connectivity of an open set G

      i. Topological properties equivalent to simple connectivity: G is homeomorphic to the unit disk; $\mathbb{C}_\infty \setminus G$ is connected (that is, G has no holes); every closed curve in G can be shrunk in G to a point (in technical language, every closed curve in G is homotopic to a constant—the fundamental group of G is trivial); every closed curve in G has zero winding number about every point in the complement of G.

      ii. Analytic properties equivalent to simple connectivity: the integral of every function analytic in G over a closed curve in G is zero; every nowhere zero analytic function in G has an analytic logarithm in G; every harmonic function in G has a harmonic conjugate in G; every analytic function in G is the

derivative of an analytic function in G; polynomials are dense in the space $H(G)$ of analytic functions in G.

iii. An algebraic property equivalent to simple connectivity: every nowhere zero analytic function in G has an analytic square root.

2. Main Theorems

(a) Runge's theorem

i. Polynomial approximation: if K is a polynomially convex compact set in $\mathbb{C}$ (that is, K has no holes), then every function analytic in a neighborhood of K can be approximated uniformly on K by polynomials; if G is a simply connected open set in $\mathbb{C}$ (that is, G has no holes), then every function analytic in G can be approximated uniformly on compact subsets of G by polynomials (that is, polynomials are dense in $H(G)$).

ii. Rational approximation: if K is a compact set in $\mathbb{C}$, then every function analytic in a neighborhood of K can be approximated uniformly on K by rational functions, and the locations of the poles may be prescribed within each component of $\mathbb{C}_\infty \setminus K$; if G is an open set in $\mathbb{C}$, then every function analytic in G can be approximated uniformly on compact subsets of G by rational functions, and the locations of the poles may be prescribed within each component of $\mathbb{C}_\infty \setminus G$.
Idea of proof: Surround K by a nice contour, write Cauchy's formula, and approximate by Riemann sums. Then push the poles around in a component of $\mathbb{C} \setminus K$ by using geometric series to approximate $1/(z-a)$ by a polynomial in $1/(z-b)$ when $|a-b|$ is small.

(b) Mittag-Leffler's theorem: A meromorphic function can be constructed with prescribed poles and prescribed singular parts in an open set G as long as the poles do not accumulate inside G.
Idea of proof: Form the sum of the singular parts, subtracting off suitable analytic functions to ensure convergence. The convergence factors can be found by Runge's theorem.