Summary of Chapter X in Conway's book

Harold P. Boas

4 April 1995

1. Main Concepts

   (a) Harmonic function

      i. A function u with continuous second partial derivatives on an open set $G \subset \mathbb{C}$ is harmonic if and only if $u_{xx} + u_{yy} \equiv 0$ in G.

      ii. A continuous function u on an open set $G \subset \mathbb{C}$ is harmonic if and only if u satisfies the mean-value property $u(a) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u(a + re^{i\theta})\, d\theta$ on each disk $\bar{B}(a; r)$ contained in G.

      iii. A (real-valued) function u on an open set $G \subset \mathbb{C}$ is harmonic if and only if u is the real part of an analytic function on each disk contained in G.

      iv. A harmonic function is infinitely differentiable.

      v. In a simply connected domain, every (real-valued) harmonic function has a harmonic conjugate: every harmonic function is the real part of an analytic function.

   (b) Subharmonic function

      i. A function ϕ having continuous second partial derivatives on an open set $G \subset \mathbb{C}$ is subharmonic if and only if $\phi_{xx} + \phi_{yy} \geq 0$ in G.

      ii. A continuous function ϕ on an open set $G \subset \mathbb{C}$ is subharmonic if and only if ϕ satisfies the subaveraging property $\phi(a) \leq \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \phi(a + re^{i\theta})\, d\theta$ on each disk $\bar{B}(a; r)$ contained in G.

iii. A continuous function ϕ is subharmonic on an open set $G \subset \mathbb{C}$ if and only if for each harmonic function u on a bounded subdomain of G, the difference $\phi - u$ is either constant or else attains no maximum in the subdomain.

iv. The subharmonic functions do not form a linear space, but the maximum of two subharmonic functions is subharmonic.

v. Modifying a subharmonic function by changing it inside a disk contained in its domain to become harmonic in the disk, while keeping it continuous at the boundary of the disk and unchanged outside the disk, preserves subharmonicity.

vi. A superharmonic function is the negative of a subharmonic function.

(c) Dirichlet problem: The problem is to find a harmonic function inside a region with prescribed values on the boundary.

(d) Poisson kernel for the unit disk

i. The solution to the Dirichlet problem for the unit disk with boundary data $f(e^{it})$ is $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, dt$, where $P_r(\theta)$ is the Poisson kernel $\frac{1 - r^2}{1 - 2r\cos\theta + r^2}$.

ii. Estimating the Poisson kernel from above and from below gives Harnack's inequality for *positive* harmonic functions in $B(a;R)$: namely, $\frac{R-r}{R+r}u(a) \le u(a + re^{i\theta}) \le \frac{R+r}{R-r}u(a)$.

(e) Perron family: This family of subharmonic functions with boundary values no larger than a given function is used to solve the Dirichlet problem in a general domain.

(f) Green function: This is a harmonic function in a punctured domain with boundary value 0 and a logarithmic singularity at the puncture.

2. Main Theorems

(a) Maximum principle

i. Local version: A nonconstant (sub)harmonic function cannot attain a local maximum at an interior point of its domain.

   ii. Global version: A (sub)harmonic function takes its maximum on the boundary of a bounded region.
   Idea of proof: By the (sub)averaging property, a maximum could not be attained at the center of a disk unless the function were equal to its average on the whole disk.

(b) Harnack's theorem: A monotonic sequence of harmonic functions converges uniformly on compact sets either to a harmonic function or to $\pm\infty$.
Idea of proof: Using the monotonicity, reduce to the case of positive harmonic functions. Then use Harnack's inequality to show that convergence at one point implies convergence in a neighborhood of the point.

(c) Solution of the Dirichlet problem: The Dirichlet problem can be solved (uniquely) in a domain having no singleton boundary component.
Idea of proof: (Uniqueness is easy from the maximum principle.) The supremum of the Perron family is always harmonic (as follows from 1(b)iv and 1(b)v on the previous page). To show that it has the correct boundary values, use a peak function or barrier to show that the Perron family does not degenerate at the boundary. A barrier can be constructed by studying the variation of the argument of a suitable logarithm that can be defined on the domain since each boundary component is a continuum.