Summary of Chapter XI in Conway's book

Harold P. Boas

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1. Main Concepts

   (a) Genus of an entire function.
   According to the Weierstrass factorization theorem, an entire function f can be written in the form

   $$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}(z/a_n),$$

   where m is a nonnegative integer, $g(z)$ is an entire function, the p_n are certain nonnegative integers chosen to make the infinite product converge, the a_n are the nonzero zeroes of f, and $E_p(z) = (1-z)\exp(z+(z^2/2)+(z^3/3)+\cdots+(z^p/p))$. If it is possible to choose all the p_n the same, then the smallest such choice, called the rank of f, is the smallest integer p such that $\sum_{n=1}^{\infty} 1/|a_n|^{p+1}$ converges. If an entire function f of rank p is written as $z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(z/a_n)$, and g is a polynomial, then the genus μ of f is the maximum of the rank p and the degree of g.

   (b) Order of an entire function

      i. The order λ of an entire function f is the infimum of those real numbers q for which there exists a radius $R(q)$ such that $|f(z)| < \exp(|z|^q)$ when $|z| > R(q)$.

      ii. Equivalently, $\lambda = \limsup_{r\to\infty} \dfrac{\log\log M(r)}{\log r}$, where $M(r)$ denotes the maximum modulus of f on the circle of radius r.

      iii. The order can be computed from the Taylor series coefficients of the entire function $\sum_{n=0}^{\infty} c_n z^n$ via $\dfrac{1}{\lambda} = \liminf_{n\to\infty} \dfrac{\log(1/|c_n|)}{n\log n}$.

iv. Examples: e^z, $\sin z$, and $\cos z$ have order 1; polynomials have order zero; and e^{e^z} has infinite order. A canonical product $\prod_{n=1}^{\infty} E_p(z/a_n)$ of rank p has order equal to the exponent of convergence of the zeroes a_n: this is the infimum of those real numbers q for which $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^q}$ converges.

2. Main Theorems

(a) Jensen's formula: A function f analytic in a neighborhood of the closed disk $\bar{B}(0;r)$ and not zero at the origin satisfies the equation

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta,$$

where the a_k are the zeroes of f in $B(0;r)$ (repeated according to multiplicity).

Idea of proof: If f has no zeroes, this is just the mean-value property for the harmonic function $\log|f|$. In the general case, divide out the zeroes of f with a Blaschke product. The same argument, starting from the Poisson integral instead of the mean-value property, yields the more general Poisson-Jensen formula:

$$\begin{aligned}&\log|f(z)| \\ &= -\sum_{k=1}^{n} \log\left|\frac{r^2 - \bar{a}_k z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log|f(re^{i\theta})|\,d\theta\end{aligned}$$

when z is not one of the zeroes of f.

(b) Hadamard's factorization theorem: The genus μ and the order λ of an entire function are related by the inequality $\mu \le \lambda \le \mu + 1$. In particular, if one of these numbers is finite, then so is the other. Therefore, an entire function of finite order admits an especially nice factorization.

Idea of proof:

i. Suppose the genus μ is finite. In the standard representation of the entire function, the factor e^g has order at most μ because g is a polynomial of degree at most μ. To estimate the growth of the canonical product, use that $\log|E_\mu(z)| \leq C|z|^{\mu+1}$ for some constant C, and that $\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\mu+1}} < \infty$.

ii. Suppose the order λ is finite. To show that f has finite rank, use Jensen's formula to show that the number of zeroes in a disk of radius r is, for sufficiently large r, less than $r^{\lambda+\epsilon}$. It follows that $\sum_{k=1}^{\infty} \frac{1}{|a_k|^{[\lambda]+1}} < \infty$, where $[\lambda]$ is the greatest integer not exceeding λ. To show that the function g in the exponential factor e^g is a polynomial of degree at most $[\lambda]$, differentiate the Poisson-Jensen formula $[\lambda]+1$ times and let the radius tend to infinity to see that the derivative $g^{([\lambda]+1)} \equiv 0$.