Summary of Chapter XII in Conway's book

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Main Theorems

1. Landau's and Bloch's Theorems.
   Consider the family of analytic functions on the unit disk normalized so that the derivative has modulus one at the origin. There is a positive radius r such that the range of every such function contains a disk of that radius; the supremum of such radii is Landau's constant. There is a positive radius r such that the range of every such function contains a schlicht (one-to-one image of an open set) disk of that radius; the supremum of such radii is Bloch's constant.
   Idea of proof: Observe that the range of a bounded analytic function on a disk contains a (schlicht) disk of size depending on the bound and on the radius of the domain. Find a point in the unit disk where $(1 - |z|)|f'(z)|$ is maximal, and apply the observation on the disk centered at this point of radius half the distance to the boundary of the unit disk.

2. Picard's little theorem.
   The range of a nonconstant entire function cannot omit two complex values.
   Idea of proof: If the nonconstant entire function f never takes the values 0 and 1, write $f = e^h$ for some entire function h. Since h never takes the values $2\pi ni$, it can be written as $i\pi(1 + \cosh 2g)$ for some entire function g. The range of g misses a lattice of points in the plane, so its range cannot contain any big disk. This contradicts Landau's theorem (or Bloch's theorem).

3. Schottky's theorem.
   Analytic functions in the unit disk that never take the values 0 and 1

satisfy uniform growth estimates.
More precisely, for each radius r less than 1 and each positive n there is a constant C such that $|f(z)| \leq C$ whenever $|z| \leq r$ and f is an analytic function on the unit disk such that $|f(0)| \leq n$ and the range of f omits the values 0 and 1.
Idea of proof: Write $f = \exp(i\pi(1 + \cosh 2g))$ as in the above construction. Since the range of g contains no large disk, it follows from Landau's theorem that g' blows up at the boundary of the disk no faster than a constant divided by the distance to the boundary. Integrating shows that g satisfies a similar growth estimate. Then f satisfies a corresponding estimate.

4. Montel-Carathéodory theorem.
The family of functions on a domain G that never take the values 0 and 1 is a normal family in $C(G; \mathbb{C}_\infty)$.
Idea of proof: It is enough to prove normality on a disk. The functions that are bounded in modulus by 1 at the center of the disk are locally bounded by Schottky's theorem, hence form a normal family. The same argument applies to the reciprocals of the remaining functions.

5. Picard's big theorem.
In every neighborhood of an essential singularity, an analytic function takes every complex value, with one possible exception, an infinite number of times.
Idea of proof: Suppose f has an essential singularity at the origin and does not take either of the values 0 and 1 an infinite number of times. Then f does not take the values 0 and 1 at all in some suitably small punctured disk. The functions $f(z/n)$ form a normal family in this domain by Montel's theorem. If a subsequence is locally bounded, then it follows that f is bounded, and hence f has a removable singularity. If a subsequence tends to infinity, then the reciprocals are locally bounded, so $1/f$ has a removable singularity, and f has a pole. In either case, a contradiction is reached with the assumption that f has an essential singularity.