Math 650-600: Several Complex Variables

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The Levi form

Suppose that (locally) Ω is $\{ z \in \mathbb{C}^n : \rho(z) < 0 \}$, where ρ is a twice-differentiable, real-valued function, and $\nabla\rho \neq 0$ where $\rho = 0$. Equivalently, Ω is locally the graph of a twice-differentiable function (by the implicit function theorem).

Theorem. Pseudoconvexity is equivalent to the property that for each boundary point a of Ω,

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(a) t_j \bar{t}_k \geq 0 \qquad \text{for every complex tangent vector } t.$$

A complex tangent vector t satisfies the equation $\displaystyle\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(a) t_j = 0$.

From pseudoconvexity to the Levi form

One choice of a local defining function is the signed distance function $\rho(z) = -\operatorname{dist}(z, b\Omega)$ when $z \in \Omega$ and $\rho(z) = +\operatorname{dist}(z, b\Omega)$ when $z \in \mathbb{C}^n \setminus \Omega$.

We know that if Ω is pseudoconvex, then $-\log(-\rho)$ is plurisubharmonic inside Ω, which means that

$$\frac{1}{|\rho|} \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k + \frac{1}{\rho^2} \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} t_j \sum_{k=1}^{n} \frac{\partial\rho}{\partial \bar{z}_k} \bar{t}_k \geq 0$$

for every vector t in $\mathbb{C}^n$.

Restricting to vectors t for which $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} t_j = 0$ gives that $\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq 0$ inside Ω, and taking the limit at the boundary shows that the Levi form is ≥ 0.

The converse direction

Suppose the Levi form is ≥ 0 on complex tangent vectors to the boundary of Ω. The goal is to show that Ω has a plurisubharmonic exhaustion function *locally*.

Choose ρ of the form $\varphi(z_1, \ldots, z_{n-1}, x_n) - y_n$ in local coordinates. Then the Levi condition

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(a) t_j \bar{t}_k \geq 0 \qquad \text{whenever } \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j = 0$$

holds for all points a in (a local coordinate patch of) Ω, not just for boundary points. We need additionally to control the size of the complex Hessian in the missing direction.

Resolve an arbitrary vector t as a sum $t' + t''$, where $\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t'_j = 0$ and $|t''| = \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j\right| / |\partial \rho|$.

Conclusion of the proof

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(a) t_j \bar{t}_k = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(a) t'_j \bar{t}'_k + O(|t| \cdot |t''|),$$

and the first term is ≥ 0 by hypothesis, so there is constant C such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(a) t_j \bar{t}_k \geq -C|t| \cdot \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j\right|.$$

The complex Hessian of $-\log(-\rho)$ applied to t is then

$$\geq \frac{-C|t|}{|\rho|} \cdot \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j\right| + \frac{1}{\rho^2} \cdot \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j\right|^2 \geq -B|t|^2$$

for some constant B (namely $B = C^2/4$).

Then $-\log(-\rho(z)) + B|z|^2$ is a local plurisubharmonic function that blows up at the boundary of Ω. Thus Ω is pseudoconvex when the Levi form of Ω is ≥ 0.

Exercises on the Levi form

1. Positivity of the Levi form is independent of the choice of defining function.

2. Use the chain rule to show that positivity of the Levi form is invariant under holomorphic changes of coordinates.
In other words, if $F = (f_1, \ldots, f_n)$ is a (local) biholomorphic mapping, then the Levi form of ρ is ≥ 0 if and only if the Levi form of $\rho \circ F$ is ≥ 0.
Part of the problem is to verify that the complex tangent space is preserved under biholomorphic mappings.