Math 650-600: Several Complex Variables

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Complete Hartogs domains

Theorem. A complete Hartogs domain

$$\{ z \in \mathbb{C}^{n+1} : |z_{n+1}| < e^{-u(z_1,\dots,z_n)}, \quad (z_1,\dots,z_n) \in G \}$$

is pseudoconvex if and only if (a) the base G is pseudoconvex and (b) the function u is plurisubharmonic on G.

First part of the proof from last time. Suppose Ω is pseudoconvex.

Then the base (or any slice of Ω by a complex hyperplane) is pseudoconvex: restrict a plurisubharmonic exhaustion function to the slice.

A previous proof shows that $-\log d_c$ is plurisubharmonic for every unit vector c, and when c points in the z_{n+1} direction, $-\log d_c(z_1,\dots,z_n,0) = u(z_1,\dots,z_n)$.

It remains to prove the converse direction of the theorem.

Proof of the converse

If the base G in $\mathbb{C}^n$ is pseudoconvex, then so is the product domain $G \times \mathbb{C}$ in $\mathbb{C}^{n+1}$. The domain Ω is cut out of the product domain by the inequality $\log|z_{n+1}| + u(z_1,\dots,z_n) < 0$, and $v(z) := \log|z_{n+1}| + u(z_1,\dots,z_n)$ is plurisubharmonic.

Lemma. If D is pseudoconvex and $\Omega = \{ z \in D : v(z) < 0 \}$, where v is plurisubharmonic in D, then Ω is pseudoconvex.

Proof. If $K \subset\subset \Omega$, then v has a negative supremum on K. If v is continuous, then the plurisubharmonic hull $\widehat{K}_\Omega$ stays away from the set where $v = 0$. Also $\widehat{K}_\Omega \subseteq \widehat{K}_D \subset\subset D$. So $\widehat{K}_\Omega \subset\subset \Omega$. Hence Ω is pseudoconvex.

If v is not continuous: by the preceding special case, $D_k := \{ z \in D : -\log \operatorname{dist}(z, bD) + |z|^2 < k \}$ is pseudoconvex and $D_k \uparrow D$; now approximate v from above on D_k by smooth plurisubharmonic functions. As $k \to \infty$, we get an exhaustion of Ω from inside by pseudoconvex domains.

Exercise on Reinhardt domains

For a complete Reinhardt domain in $\mathbb{C}^2$ with smooth boundary, show that the Levi form is ≥ 0 if and only if the domain is logarithmically convex.

This solves a special case of the Levi problem.

Held over for next time.

Exercise on tube domains

An unbounded domain Ω in $\mathbb{C}^n$ is called a tube domain with base G in $\mathbb{R}^n$ if $\Omega = \{\, x + iy \in \mathbb{C}^n : x \in G \text{ and } y \in \mathbb{R}^n \,\}$.

Exercise. Show that a tube domain in $\mathbb{C}^n$ is pseudoconvex if and only if the base G in $\mathbb{R}^n$ is convex.

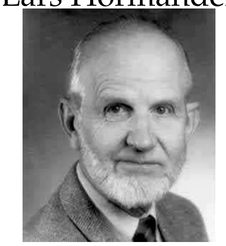
Held over for next time.

Solvability of the $\bar{\partial}$-equation

Theorem. Suppose Ω is a pseudoconvex domain in $\mathbb{C}^n$. Then for every $\bar{\partial}$-closed $(0,1)$-form f with coefficients of class $C^\infty(\Omega)$, there exists a function u of class $C^\infty(\Omega)$ such that $\bar{\partial}u = f$.

We will prove the theorem using Hörmander's L^2 method, but first we will apply the theorem to solve the Levi problem.

Lars Hörmander



born 24 Jan 1931

Hörmander won the 1962 Fields Medal for fundamental work on linear partial differential equations.

Extension from slices

Theorem. Suppose Ω is a pseudoconvex domain in $\mathbb{C}^n$, and let ω be the intersection of Ω with a complex hyperplane. For every holomorphic function f on ω (in $\mathbb{C}^{n-1}$), there is a holomorphic function F on Ω such that $F|_\omega = f$.

Proof. We may assume that the slice is given by $z_n = 0$. The set ω is relatively closed in Ω, and so is the set $\Omega \setminus (\omega \times \mathbb{C})$. These sets are disjoint, so there is a class C^∞ function φ on Ω such that $\varphi \equiv 1$ on ω and $\varphi \equiv 0$ on $\Omega \setminus (\omega \times \mathbb{C})$. Then φf makes sense as a class C^∞ function on Ω.

We seek F in the form $\varphi f - z_n v$, with v to be determined. To make F holomorphic, we need to have $\bar{\partial}v = \frac{1}{z_n}\bar{\partial}(\varphi f)$. By construction, the right-hand side is class C^∞ on Ω and $\bar{\partial}$-closed. By the theorem on solvability of the $\bar{\partial}$-equation on pseudoconvex domains, the required function v exists.