Math 650-600: Several Complex Variables

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Recap from last time

We reduced the solution of the $\bar\partial$-equation on $(0,1)$-forms to the following.

Key estimate. If Ω is a pseudoconvex domain in $\mathbb{C}^n$, then there exist smooth weight functions φ_1, φ_2, and φ_3 such that when $f = \sum_{j=1}^n f_j\, d\bar z_j$ is in the intersection of the domains of $\bar\partial$ and $\bar\partial^*$, then

$$\|f\|_{\varphi_2} \le \|\bar\partial^* f\|_{\varphi_1} + \|\bar\partial f\|_{\varphi_3}.$$

To prove the key estimate, we need (a) to compute the terms on the right-hand side and (b) to choose suitable weight functions.

Computation of the norm of $\bar\partial f$

A clever technical point: if the weight functions φ_1, φ_2, and φ_3 are chosen to grow suitably fast at $b\Omega$, then the C^∞ functions with compact support in Ω will be dense in $\operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*$. Assuming this density for now, we may compute in $C_0^\infty(\Omega)$.

If $f = \sum_{j=1}^n f_j\, d\bar z_j$, then $\bar\partial f = \sum_{j<k} \left(\frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j}\right) d\bar z_k \wedge d\bar z_j$, so $|\bar\partial f|^2 = \sum_{j<k} \left|\frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j}\right|^2 = \frac{1}{2}\sum_{j,k} \left|\frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j}\right|^2 = \sum_{j,k}\left(\left|\frac{\partial f_j}{\partial \bar z_k}\right|^2 - \frac{\partial f_j}{\partial \bar z_k}\overline{\frac{\partial f_k}{\partial \bar z_j}}\right)$. Therefore

$$\|\bar\partial f\|_{\varphi_3}^2 = \sum_{j,k} \int_\Omega \left(\left|\frac{\partial f_j}{\partial \bar z_k}\right|^2 - \frac{\partial f_j}{\partial \bar z_k}\,\overline{\frac{\partial f_k}{\partial \bar z_j}} \right) e^{-\varphi_3}$$

and we need to handle the term with the minus sign.

Computation of $\overline{\partial}^*$

The choice of weight functions (to be made later) guarantees that $C_0^\infty(\Omega)$ is dense in $\operatorname{Dom}\overline{\partial}$, so to compute $\overline{\partial}^*$ we may pair with a smooth, compactly supported function and integrate by parts without boundary terms.

$\langle\overline{\partial}^* f, g\rangle_{\varphi_1} = \langle f, \overline{\partial} g\rangle_{\varphi_2} = \sum_{j=1}^{n}\langle f_j, \frac{\partial g}{\partial \bar z_j}\rangle_{\varphi_2} = -\sum_{j=1}^{n}\langle e^{\varphi_1}\frac{\partial}{\partial z_j}(e^{-\varphi_2} f_j), g\rangle_{\varphi_1}$.

So $\overline{\partial}^* f = -\sum_{j=1}^{n} e^{\varphi_1}\frac{\partial}{\partial z_j}(e^{-\varphi_2} f_j)$. Set $\delta_j f_j := e^{\varphi_3}\frac{\partial}{\partial z_j}(e^{-\varphi_3} f_j)$ and rewrite $\overline{\partial}^* f = -e^{\varphi_1-\varphi_2}\sum_{j=1}^{n}\big(\delta_j f_j - f_j\frac{\partial}{\partial z_j}(\varphi_2-\varphi_3)\big)$.

We do not need three independent weight functions. It is convenient to use two independent functions $\varphi := \varphi_3$ and ψ satisfying $\varphi_3-\varphi_2=\psi=\varphi_2-\varphi_1$. Then $\overline{\partial}^* f = -e^{-\psi}\sum_{j=1}^{n}\left(\delta_j f_j + f_j\frac{\partial\psi}{\partial z_j}\right)$.

Approaching the key estimate

By the triangle inequality, $\|e^{-\psi}\sum_{j=1}^{n}\delta_j f_j\|_{\varphi_1} \le \|\overline{\partial}^* f\|_{\varphi_1} + \|e^{-\psi}\sum_{j=1}^{n} f_j\frac{\partial\psi}{\partial z_j}\|_{\varphi_1}$.

Since $2\psi+\varphi_1=\varphi_3=\varphi$, squaring both sides gives

$\int_\Omega \big|\sum_{j=1}^{n}\delta_j f_j\big|^2 e^{-\varphi} \le 2\|\overline{\partial}^* f\|^2_{\varphi_1} + 2\int_\Omega |f|^2\,|\partial\psi|^2\,e^{-\varphi}$.

Adding the previously computed term $\|\overline{\partial} f\|^2_{\varphi_3}$ yields

$$\sum_{j,k}\int_\Omega\left((\delta_j f_j)(\overline{\delta_k f_k}) - \frac{\partial f_j}{\partial \bar z_k}\overline{\frac{\partial f_k}{\partial \bar z_j}}\right)e^{-\varphi} \le \|\overline{\partial} f\|^2_{\varphi_3} + 2\|\overline{\partial}^* f\|^2_{\varphi_1} + 2\int_\Omega |f|^2\,|\partial\psi|^2\,e^{-\varphi}.$$

Because δ_j is adjoint to $-\partial/\partial\bar z_j$ on $C_0^\infty(\Omega)$, the left-hand side simplifies after two integrations by parts.

The punch line

The commutator $\delta_j \frac{\partial}{\partial \bar{z}_k} - \frac{\partial}{\partial \bar{z}_k}\delta_j = \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}$. Therefore

$$\int_\Omega \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f_j \overline{f_k}\, e^{-\varphi} \le \|\bar{\partial} f\|_{\varphi_3}^2 + 2\|\bar{\partial}^* f\|_{\varphi_1}^2 + 2\int_\Omega |f|^2\, |\partial \psi|^2\, e^{-\varphi}.$$

If we can choose φ sufficiently strongly plurisubharmonic that for every vector w in $\mathbb{C}^n$ we have

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \ge 2|w|^2\, |\partial \psi|^2 + 2|w|^2\, e^{\psi}$$

then we will have the key estimate, and hence the solution of the $\bar{\partial}$-equation on pseudoconvex domains.