Math 650-600: Several Complex Variables

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Polynomial approximation

Mergelyan's theorem in the plane. If K is compact and $\mathbb{C} \setminus K$ is connected, then every continuous function on K that is holomorphic in the interior of K can be approximated uniformly on K by holomorphic polynomials.

Exercise. The conclusion of Mergelyan's theorem holds on the bidisc in $\mathbb{C}^2$.

Exercise. The conclusion of Mergelyan's theorem does not hold on the Hartogs triangle in $\mathbb{C}^2$.

The $\overline{\partial}$-problem

If $\overline{\partial} g = 0$, does there exist f such that $\overline{\partial} f = g$?

The problem is solvable *locally*, but whether a *global* solution exists depends on the domain (when $n \geq 2$).

Solution of the $\overline{\partial}$-problem in the plane. If g has continuous first partial derivatives on the closure of a bounded domain G in $\mathbb{C}$, then a solution of the equation $\frac{\partial f}{\partial \bar{z}} = g$ is given by

$$f(z) = \frac{1}{2\pi i} \int_G \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

We will see that the proof follows from Cauchy's formula with remainder.

Cauchy's formula with remainder

If Ω is a bounded domain in $\mathbb{C}$ whose boundary is a continuously differentiable curve γ, and if the function g has continuous first partial derivatives on the closure of Ω, then for every point z in Ω we have the integral representation

$$g(z) = \frac{1}{2\pi i} \oint_\gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_\Omega \frac{\partial g/\partial \overline{\zeta}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}.$$

Proof. Apply the theorem of Green/Stokes to $\Omega \setminus B(z, \epsilon)$ and let $\epsilon \to 0$.
Observe that
$$\frac{1}{2\pi i} \oint_{\partial B(z,\epsilon)} \frac{g(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} g(z + \epsilon e^{i\theta})\, d\theta \xrightarrow{\epsilon \to 0} g(z).$$

Green and Stokes

George Green
(1793–1841)

Self-taught mathematician

He introduced the mathematical term "potential function".

George Gabriel Stokes

(1819–1903)
Mathematician and physicist

Solving $\overline{\partial}$ in the plane: proof

First suppose that g has compact support in the domain G. Then
$f(z) := \frac{1}{2\pi i}\int_G \frac{g(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\zeta + z)}{\zeta}\,d\zeta \wedge d\overline{\zeta}$, so $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g/\partial\overline{\zeta}}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}$.

The right-hand side is precisely $g(z)$ by the Cauchy integral formula with remainder written for a large disc containing the support of g.

Final step: if g does not have compact support, take a bump function φ supported near some z_0 and identically equal to 1 in a neighborhood of z_0. Write $f(z)$ as the sum of $\frac{1}{2\pi i}\int_G \frac{\varphi(\zeta)g(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}$ and $\frac{1}{2\pi i}\int_G \frac{(1-\varphi(\zeta))g(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}$.

Exercises on $\overline{\partial}$ in the plane

1. Find an explicit (smooth) solution to the equation $\partial f/\partial\overline{z} = 1/z$ in the punctured plane $\mathbb{C} \setminus \{0\}$.
2. Find an explicit (smooth) solution to the equation $\partial f/\partial\overline{z} = 1/\overline{z}$ in the punctured plane $\mathbb{C} \setminus \{0\}$.