Math 650-600: Several Complex Variables

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Exercises on convexity

A domain Ω in $\mathbb{C}^n$ is convex with respect to a set $\mathcal{F}$ of real-valued functions on Ω if $K \subset\subset \Omega \Rightarrow \widehat{K}_{\mathcal{F}} \subset\subset \Omega$.
Here $\widehat{K}_{\mathcal{F}}$ denotes the $\mathcal{F}$-hull of K: the set of points z in Ω such that $f(z) \leq \sup\{f(w) : w \in K\}$ for every function f in $\mathcal{F}$.
The notation "$K \subset\subset \Omega$" means that K is a "relatively compact" subset of Ω: the closure of K is a compact subset of Ω.

Exercise. Let $\mathcal{F}$ be the set of real parts of holomorphic polynomials of degree 1. Show that Ω is convex with respect to $\mathcal{F}$ if and only if Ω is convex in the ordinary geometric sense.

Exercise. Let $\mathcal{F}$ be the set of moduli of holomorphic polynomials of degree 1. Show that Ω is convex with respect to $\mathcal{F}$ if and only if Ω is convex in the ordinary geometric sense.

Polynomial convexity

When $\mathcal{F}$ is the set of moduli of holomorphic polynomials, convexity with respect to $\mathcal{F}$ is called *polynomial convexity*.

A version of Runge's theorem. A domain Ω in the plane $\mathbb{C}$ is polynomially convex if and only if Ω is simply connected.

When $n \geq 2$, there is no topological characterization of polynomial convexity.

Example/theorem (Eva Kallin, 1964). There exist three disjoint closed polydiscs in $\mathbb{C}^3$ whose union is not polynomially convex. On the other hand, the union of three disjoint closed balls in $\mathbb{C}^n$ is always polynomially convex.

Open problem. Is the union of four disjoint closed balls in $\mathbb{C}^n$ always polynomially convex?

Linear (fractional) convexity

Let Ω be a domain in $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of moduli of linear fractional functions $\big(a + \sum_{j=1}^{n} b_j z_j\big) / \big(c + \sum_{j=1}^{n} d_j z_j\big)$ that are holomorphic on Ω.

Theorem. A necessary and sufficient condition for Ω to be convex with respect to $\mathcal{F}$ is that for each point p of the boundary of Ω there is a complex hyperplane that passes through p and does not intersect Ω.

A domain with this property is called *weakly linearly convex*.

Exercise. Make a Venn diagram showing the relationships among the following concepts:

- convexity
- holomorphic convexity
- polynomial convexity
- weak linear convexity

Proof of the theorem

If Ω has a supporting complex hyperplane at a boundary point p, then the reciprocal of the defining equation of this hyperplane is a linear fractional function whose modulus belongs to $\mathcal{F}$ and which blows up at p, so the $\mathcal{F}$-hull of a compact set K stays away from p.

Conversely, if Ω is convex with respect to the set $\mathcal{F}$ of moduli of linear fractional functions, and p is a boundary point of Ω, take an exhaustion of Ω by nested $\mathcal{F}$-convex compact sets K_j and a sequence of points p_j such that $p_j \to p$ and $p_j \notin K_j$.

There is a linear fractional function f_j such that $\max\{|f_j(z)| : z \in K_j\} < 1$ and $f_j(p_j) = 1$. The level set $\{\, z : f_j(z) = 1 \,\}$ is a complex hyperplane passing through p_j. Passing to a subsequence, we find a limiting complex hyperplane passing through p which does not intersect Ω.

Two references

Mats Andersson, Mikael Passare, and Ragnar Sigurdsson, *Complex convexity and analytic functionals*, Birkhäuser, 2004; QA639.5 .A53 2004.

Lars Hörmander, *Notions of convexity*, Birkhäuser, 1994; QA639.5 .H67 1994.