Math 650-600: Several Complex Variables

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Exercises on pseudoconvexity

Without using the solution of the Levi problem, show that pseudoconvexity is preserved under the following set operations.

1. Taking Cartesian products: because a compact set K is contained in a product $K_1 \times K_2$, and $\widehat{K_1 \times K_2} \subseteq \widehat{K_1} \times \widehat{K_2}$.

2. Taking increasing unions: use version II of the continuity principle; or realize $-\log \operatorname{dist}(z, b\Omega)$ as a decreasing limit of plurisubharmonic functions.

3. Taking a connected component of the interior of an intersection: realize $-\log \operatorname{dist}(z, b\Omega)$ as $\sup_\alpha(-\log \operatorname{dist}(z, b\Omega_\alpha))$.

Pseudoconvexity is a local property

Theorem. If every boundary point of a domain Ω in $\mathbb{C}^n$ has a neighborhood U such that every component of $U \cap \Omega$ is pseudoconvex, then Ω is pseudoconvex.

Proof. If B is a ball contained in U centered at a boundary point of Ω, and z is in a concentric ball of half the radius, then $\operatorname{dist}(z, b\Omega) = \operatorname{dist}(z, b(U \cap \Omega))$.

Consequently, if $z \in \Omega_j$ (where Ω_j is a certain component of $U \cap \Omega$), then $\operatorname{dist}(z, b\Omega) = \operatorname{dist}(z, b\Omega_j)$, so the plurisubharmonicity of $-\log \operatorname{dist}(z, b\Omega_j)$ implies the plurisubharmonicity of $-\log \operatorname{dist}(z, b\Omega)$ in a neighborhood of the boundary of Ω.

To get a plurisubharmonic exhaustion function for Ω, take $\max(-\log \operatorname{dist}(z, b\Omega), \varphi(z))$, where φ is a large constant if Ω is bounded, and φ is a rapidly increasing convex function of $|z|^2$ if Ω is unbounded.

The Levi form

Suppose that a piece of the boundary of Ω is a class C^2 real hypersurface; equivalently, there is a local real-valued *defining function* ρ such that $\rho < 0$ inside Ω, on the boundary $\rho = 0$, and $\rho > 0$ outside the closure of Ω; and ρ is twice continuously differentiable, and the gradient of ρ is not equal to zero anywhere on the boundary of Ω.

Theorem. Pseudoconvexity is equivalent to a local differential-geometric property of the boundary of Ω: namely, for each boundary point a,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(a) t_j \bar{t}_k \geq 0 \qquad \text{whenever } \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a) t_j = 0.$$

The condition does *not* mean that ρ is plurisubharmonic, for the inequality holds only when t is a *complex tangent vector*.