Goncarov-Type Polynomials and Applications in Combinatorics

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Abstract

In this paper we extend the work of [1] to study combinatorial problems via the theory of biorthogonal polynomials. In particular, we give a unified algebraic approach to several combinatorial objects, including order statistics of a real sequence, parking functions, lattice paths, and area-enumerators of lattice paths, by describing the properties of the sequence of Goncarov polynomials and its various generalizations.

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1 Introduction

The main content of this paper is to use theory of sequences of polynomials biorthogonal to a sequence of linear operators to study combinatorial problems. In particular, we described the algebraic properties of the sequence of Goncarov polynomials and its various generalizations, which give a unified algebraic approach to several combinatorial objects, including (1) The cumulative distribution functions of the random vectors of order statistics of n independent random variables with uniform distribution on an interval; (2) general parking functions, that is, sequences $(x_1, x_2, \ldots, x_n)$ of integers whose order statistics are bounded between two given non-decreasing sequences; (3) Lattice paths that avoid certain general boundaries; and (4) The area-enumerator of lattice paths avoiding certain general boundaries. The object (2) can be viewed as a discrete analog of (1). In literature, objects (1) and (3) have been extensively studied by probabilistic argument and counting techniques. General parking functions with one boundary has been studied in a previous paper by the first and the third author [2].

The contribution of the current paper is to put all four problems in the same umbrella, and present a unified treatment. For object (1) and (2), the corresponding polynomial sequences is Goncarov polynomials, which are outlined in Section 2. This section also contains an introduction to the theory of sequences of biorthogonal polynomials. In Section 3 we describe the sequences of difference Goncarov polynomials. The combinatorial interpretation of difference Goncarov polynomials is lattice paths with one-sided boundary, which is given in Section 4. Section 5 and 6 are on q-analog of difference Goncarov polynomials, and its application in enumerating area of lattices paths with one-sided boundary. The two-sided boundaries for both parking functions and lattice paths are treated in Section 7.

2 Sequences of biorthogonal polynomials and Goncarov polynomials

We begin by giving an outline of the theory of sequences of polynomials biorthogonal to a sequence of linear functionals. The details can be found in [1].

Let $\mathcal{P}$ be vector space of all polynomials in the variable x over a field F of characteristic zero. Let $D : \mathcal{P} \to \mathcal{P}$ be the differentiation operator. For a scalar a in the field F, let

$$\varepsilon(a) : \mathcal{P} \to F,\ p(x) \mapsto p(a)$$

be the linear functional which evaluates $p(x)$ at a.

Let $\varphi_s(D), s = 0, 1, 2, \ldots$ be a sequence of linear operators on $\mathcal{P}$ of the form

$$\varphi_s(D) = D^s \sum_{r=0}^{\infty} b_{sr} D^r, \tag{2.1}$$

where the coefficients b_{s0} are assumed to be non-zero. There exists a unique sequence $p_n(x), n = 0, 1, 2, \ldots$ of polynomials such that $p_n(x)$ has degree n and

$$\varepsilon(0)\varphi_s(D)p_n(x) = n!\delta_{sn}, \tag{2.2}$$

where δ_{sn} is the Kronecker delta.

The polynomial sequence $p_n(x)$ is said to be *biorthogonal* to the sequence $\varphi_s(D)$ of operators, or, the sequence $\varepsilon(0)\varphi_s(D)$ of linear functionals. Using Cramer's rule to solve the linear system and Laplace's expansion to group the results, we can express $p_n(x)$ by the the following *determinantal formula*:

$$p_n(x) = \frac{n!}{b_{00}b_{10}\cdots b_{n0}} \begin{vmatrix} b_{00} & b_{01} & b_{02} & \ldots & b_{0,n-1} & b_{0n} \\ 0 & b_{10} & b_{11} & \ldots & b_{1,n-2} & b_{1,n-1} \\ 0 & 0 & b_{20} & \ldots & b_{2,n-3} & b_{2,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-1,0} & b_{n-1,1} \\ 1 & x & x^2/2! & \ldots & x^{n-1}/(n-1)! & x^n/n! \end{vmatrix}. \tag{2.3}$$

Since $\{p_n(x)\}_{n=0}^{\infty}$ forms a basis of $\mathcal{P}$, any polynomial can be uniquely expressed as a linear combination of $p_n(x)$'s. Explicitly, we have the *expansion formula*: If $p(x)$ is a polynomial of degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{d_i p_i(x)}{i!}, \tag{2.4}$$

where $d_i = \varepsilon(0)\varphi_i(D)p(x)$. In particular,

$$x^n = \sum_{i=0}^{n} \frac{n! b_{i,n-i} p_i(x)}{i!}, \tag{2.5}$$

which gives a *linear recursion* for $p_n(x)$. Equivalently, one can write (2.5) in terms of formal power series equations, and obtain the *Appell relation*

$$e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)\varphi_n(t)}{n!}, \tag{2.6}$$

where $\varphi_n(t) = t^s \sum_{r=0}^{\infty} b_{sr} t^r$.

A special example of sequences of biorthogonal polynomials is the Goncarov polynomials. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of numbers or variables called *nodes.* The sequence of *Gončarov polynomials*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}),\ \ n = 0, 1, 2, \ldots$$

is the sequence of polynomials biorthogonal to the operators

$$\varphi_S(D) = D^s \sum_{r=0}^{\infty} \frac{a_s^r D^r}{r!}.$$

As indicated by the notation, $g_n(x; a_0, a_1, \ldots, a_{n-1})$ depends only on the nodes $a_0, a_1, \ldots, a_{n-1}$. Indeed, from equation (2.3), we have the *determinantal formula,*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n! \begin{vmatrix} 1 & a_0 & \frac{a_0^2}{2!} & \frac{a_0^3}{3!} & \cdots & \frac{a_0^{n-1}}{(n-1)!} & \frac{a_0^n}{n!} \\ 0 & 1 & a_1 & \frac{a_1^2}{2!} & \cdots & \frac{a_1^{n-2}}{(n-2)!} & \frac{a_1^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & a_2 & \cdots & \frac{a_2^{n-3}}{(n-3)!} & \frac{a_2^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{n-1} \\ 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \end{vmatrix}.$$

From equations (2.5) and (2.6), we have the *linear recursion*

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1})$$

and the *Appell relation*

$$e^{xt} = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_{n-1}) \frac{t^n e^{a_n t}}{n!}.$$

Finally, from equation (2.4), we have the *expansion formula.* If $p(x)$ is a polynomial of degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(a_i) D^i p(x)}{i!} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

The sequence of Goncarov polynomials possesses a set of specific properties, which are listed in the following. The proofs can be found in [1].

1. *Differential relations.*

   The Gončarov polynomials can be equivalently defined by the differential relations

$$Dg_n(x; a_0, a_1, \ldots, a_{n-1}) = ng_{n-1}(x; a_1, a_2, \ldots, a_{n-1}),$$

   with initial conditions

$$g_n(a_0; a_0, a_1, \ldots, a_{n-1}) = \delta_{0n}.$$

2. *Integral relations.*

$$\begin{aligned} g_n(x; a_0, a_1, \ldots, a_{n-1}) &= n \int_{a_0}^{x} g_{n-1}(t; a_1, a_2, \ldots, a_{n-1}) dt \\ &= n! \int_{a_0}^{x} dt_1 \int_{a_1}^{t_1} dt_2 \cdots \int_{a_{n-1}}^{t_{n-1}} dt_n. \end{aligned}$$

3. *Shift invariant formula.*

$$g_n(x + \xi; a_0 + \xi, a_1 + \xi, \ldots, a_{n-1} + \xi) = g_n(x; a_0, a_1, \ldots, a_{n-1}).$$

4. Perturbation formula.

$$\begin{aligned} & g_n(x; a_0, \ldots, a_{m-1}, a_m + b_m, a_{m+1}, \ldots, a_{n-1}) = g_n(x; a_0, \ldots a_{m-1}, a_m, a_{m+1}, \ldots, a_{n-1}) \\ - & \binom{n}{m} g_{n-m}(a_m + b_m; a_m, a_{m+1}, \ldots, a_{n-1}) g_m(x; a_0, a_1, \ldots, a_{m-1}). \end{aligned}$$

   Applying the perturbation formula repeatedly, we can perturb any subset of nodes. For example, the following formula allows us to perturb an initial segment of length $n - m + 1$:

$$\begin{aligned} & g_n(x; a_0 + b_0, a_1 + b_1, \ldots, a_{n-m} + b_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\ = \; & g_n(x; a_0, a_1, \ldots, a_{n-m}, a_{n-m+1}, \ldots, a_{n-1}) \\ & - \sum_{i=0}^{n-m} \binom{n}{i} g_{n-i}(a_i + b_i; a_i, a_{i+1}, \ldots, a_{n-1}) g_i(x; a_0 + b_0, a_1 + b_1, \ldots, a_{i-1} + b_{i-1}). \end{aligned}$$

5. Binomial expansion.

$$g_n(x + y; a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, \ldots, a_{n-1}) x^i.$$

In particular,

$$g_n(x; a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0, a_i, \ldots, a_{n-1}) x^i.$$

That is, coefficients of Goncarov polynomials are constant terms of (shifted) Goncarov polynomials.

6. *Combinatorial representation.* Let $\mathbf{u}$ be a sequence of non-decreasing positive integers. A $\mathbf{u}$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ whose order statistics (the sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ obtained by rearranging the original sequence in non-decreasing order) satisfy $x_{(i)} \leq u_i$.

   Goncarov polynomials form a natural basis of polynomials for working with $\mathbf{u}$-parking functions. Explicitly, we have

$$\begin{aligned} P_n(u_1, u_2, \ldots, u_n) &= g_n(x; x - u_1, x - u_2, \ldots, x - u_n) \\ &= g_n(0; -u_1, -u_2, \ldots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \end{aligned}$$

For more properties and computations of parking functions via Goncarov polynomials, please refer to [1, 2, 3]. In particular, the sum enumerator and factorial moments of the sums are computed. For $\mathbf{u}$-parking functions, the sum enumerator is a specialization of $g_n(x; a_0, a_1, \ldots, a_{n-1})$ with $a_i = 1 + q + \cdots + q^{u_i - 1}$. Generating functions for factorial moments of sums of $\mathbf{u}$-parking functions are given in [1], while the explicit formulas for the first and second factorial moments of sums of $\mathbf{u}$-parking functions are given in [2], and in [3] for all factorial moments for classical parking functions where u_i forms an arithmetic progression.

Remark. Sequences of polynomials of binomial type and the related Sheffer sequences can be viewed as special cases of sequences of biorthogonal polynomials. We shall use a description given in the classical paper of Rota, Kahaner and Odlyzko [9]. A delta operator B is a formal power series of order 1 in the derivative operator D,

$$B(D) = D + b_2 D^2 + b_3 D^3 + \cdots$$

A Sheffer sequence $\{s_n\}$ (for B) is a polynomial sequence such that

$$B s_n = s_{n-1}$$

for all $n = 0, 1, 2, \ldots,$ The basic sequence $\{b_n\}$ (for B) is the Sheffer sequence with initial values $b_n(0) = \delta_{0,n}$. Basic sequence is also called sequences of binomial type, which has generating function of the form

$$\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = e^{x f(t)}, \tag{2.7}$$

where $f(t)$ is the compositional inverse of $B(x) = x + b_2x^2 + b_3x^3 + \cdots$. Sheffer sequences have generating functions of the form

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{s(t)}e^{xf(t)}, \tag{2.8}$$

where $f(t)$ is as above, and $s(t) = \sum_{n\geq 0} s_n(0)t^n$ is a formal power series of order 0. Substituting $B(t)$ for t in (2.8), we obtain the Appell relation

$$e^{xt} = \sum_{n=0}^{\infty} s_n(x)\frac{s(B(t))[B(t)]^n}{n!}.$$

From this we conclude that Sheffer polynomials can be viewed as sequences of polynomials biorthogonal to operator sequences of the form

$$\varphi_s(D) = s(B(D))[B(D)]^n,$$

where $B(t), s(t)$ are formal power series with $s(0) \neq 0$, $B(0) = 0$ and $B'(0) \neq 0$.

It is known that Sheffer sequences with special initial values can be used to study lattice path enumeration and empirical distribution functions, where the corresponding delta operators are D and the backward difference operator Δ. See [5, 6] and their references. For example, in studying the order statistics of a set of uniformly distributed random variables in $[0, 1]$, let $s_n(x) := g_n(x; a_n, a_{n-1}, \ldots, a_1)$. Since $Ds_n(x) = ns_{n-1}(x)$, we get a Sheffer sequence $\{s_n\}$ with initial values $s_n(a_n) = \delta_{0,n}$. Hence computing the empirical distribution is reduced to compute Sheffer polynomials with given initial values. For lattice path enumeration, one just replace D with Δ, (See Section 3 and 4 for details). Niederhausen has used Umbral Calculus to find explicit solutions for lattices paths in the following cases: (1) the boundary points a_n are piecewise affine in n, (2) the steps are in several directions, and (3) lattice paths are weighted by the number of left turns. The Sheffer sequence is also used to enumerate lattice paths inside a band parallel to the diagonal, which is a special case described in Section 7.

In this paper, we use the framework of sequences of biorthogonal polynomials for the following reason: (1) It is more general, while almost all the nice formulas for Sheffer sequences can be extended to this general setting, (2) It is a natural algebraic correspondence for working with parking functions and lattice paths, by the combinatorial decomposition theorem for parking functions [1, Theorem5.1], and its analog in lattice paths (c.f. Section 4). And (3). The theory of biorthogonal polynomials gives a unified treatment to several combinatorial objects simultaneously, including parking functions, order statistics of a set of uniformly distributed random variables, lattice paths, and the area-enumerator of lattice paths.

3 Difference Goncarov Polynomial

In this section we discuss the difference analog of Goncarov polynomials, which is the sequence of polynomials biorthogonal to a sequence of linear operators defined by formal power series of the *(backward) difference operators* Δ. Explicitly, let $p(x)$ be a polynomial in the vector space $\mathcal{P} = F[x]$. Define

$$\Delta p(x) = p(x) - p(x-1).$$

Note that $\Delta p(x)$ is a polynomial of x whose degree is one less than that of $p(x)$.

We follow the convention that the upper factorial $x^{(n)}$ is $x(x+1)\cdots(x+n-1)$. Observe that the polynomials $p_n(x) = x^{(n)}$ form a basis of the vector space $\mathcal{P}$; $\Delta p_n(x) = np_{n-1}(x)$; and $\Delta^i p_n(x)\big|_{x=0} = 0$, whenever $i < n$. Given a sequence $b_0, b_1, \ldots,$ let $\psi_S(\Delta)$, $s = 0, 1, 2, \ldots$ be the linear operators

$$\psi_s(\Delta) = \sum_{r=0}^{\infty} \frac{b_s^{(r)}}{r!} \Delta^{r+s}. \tag{3.1}$$

The difference Goncarov polynomials

$$\tilde{g}_n(x; b_0, \ldots, b_{n-1}), n = 0, 1, 2, \ldots$$

is the the unique sequence of polynomials satisfying $\deg(\tilde{g}_n(x; b_0, \ldots, b_{n-1})) = n$ and

$$\psi_s(\Delta)\tilde{g}_n(x; b_0, \ldots, b_{n-1})|_{x=0} = n!\delta_{sn}.$$

Many properties of Goncarov polynomials have a difference analog. which are listed in the following list. Most proofs are similar to that of the differential case, and hence omitted or only given a sketch.

1. Determinantal formula.

$$\tilde{g}_n(x; b_0, \ldots, b_{n-1}) = n! \begin{vmatrix} 1 & b_0 & \frac{b_0^{(2)}}{2!} & \frac{b_0^{(3)}}{3!} & \cdots & \frac{b_0^{(n-1)}}{(n-1)!} & \frac{b_0^{(n)}}{n!} \\ 0 & 1 & b_1 & \frac{b_1^{(2)}}{2!} & \cdots & \frac{b_1^{(n-2)}}{(n-2)!} & \frac{b_1^{(n-1)}}{(n-1)!} \\ 0 & 0 & 1 & b_2 & \cdots & \frac{b_2^{(n-3)}}{(n-3)!} & \frac{b_2^{(n-1)}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & b_{n-1} \\ 1 & x & \frac{x^{(2)}}{2!} & \frac{x^{(3)}}{3!} & \cdots & \frac{x^{(n-1)}}{(n-1)!} & \frac{x^{(n)}}{n!} \end{vmatrix}. \tag{3.2}$$

2. Expansion formula. If $p(x)$ is a polynomial of degree n, then

$$p(x) = \sum_{i=0}^{n} \frac{\psi_i(\Delta)(p(x))|_{x=0}}{i!} \tilde{g}_i(x; b_0, \dots, b_{i-1}). \tag{3.3}$$

It is obtained by applying D_i on both sides and then setting $x = 0$.

3. Linear recurrence. Let $p(x) = x^{(n)}$ in (3.3), we get

$$x^{(n)} = \sum_{i=0}^{n} \binom{n}{i} b_i^{(n-i)} \tilde{g}_i(x; b_0, \dots, b_{i-1}). \tag{3.4}$$

4. Appell relation.

$$(1-t)^{-x} = \sum_{n=0}^{\infty} \tilde{g}_n(x; b_0, \dots, b_{n-1}) \frac{t^n}{(1-t)^{b_n} n!}.$$

5. Difference relation.

$$\Delta \tilde{g}_n(x; b_0, \dots, b_{n-1}) = n \tilde{g}_{n-1}(x; b_1, \dots, b_{n-1}), \tag{3.5}$$

and

$$\tilde{g}_n(b_0; b_0, \dots, b_{n-1}) = \delta_{0n}. \tag{3.6}$$

The above difference relation and initial condition uniquely determine the sequence of difference Goncarov polynomials.

6. Summation formula. When x, b_0 are integers, solving the difference relation, we have the summation

$$\tilde{g}_n(x; b_0, b_1, \dots, b_{n-1}) = n \sum_{t=b_0+1}^{x} \tilde{g}_{n-1}(t; b_1, \dots, b_{n-1}). \tag{3.7}$$

Iterating this when $x, b_i \in \mathbb{Z}$, we have the summation formula

$$\tilde{g}_n(x; b_0, \dots, b_{n-1}) = n! \sum_{i_1=b_0+1}^{x} \left(\sum_{i_2=b_1+1}^{i_1} \left(\cdots \sum_{i_n=b_{n-1}+1}^{i_{n-1}} 1 \right) \right), \tag{3.8}$$

where

$$\sum_{i=w_1}^{w_2} \alpha(i) = \begin{cases} \alpha(w_1) + \alpha(w_1+1) + \cdots + \alpha(w_2) & \text{if} \quad w_1 \le w_2; \\ 0 & \text{if} \quad w_1 = w_2 + 1; \\ -\alpha(w_2+1) - \alpha(w_2+2) - \cdots - \alpha(w_1 - 1) & \text{if} \quad w_1 > w_2 + 1. \end{cases} \tag{3.9}$$

7. Shift-invariant formula. Using a change of variable, the summation relation (3.7), and induction, one obtain the following shift-invariant formula

$$\tilde{g}_n(x+t; b_0+t, \ldots, b_{n-1}+t) = \tilde{g}_n(x; b_0, \ldots, b_{n-1}). \tag{3.10}$$

Note that Equation (3.10) holds for all x, t, and b_i's, since it is a polynomial identity which is true for infinitely many values of x, t and b_i's.

8. Perturbation formula.

$$\begin{aligned} &\tilde{g}_n(x; b_0, \ldots, b_{m-1}, b_m+\delta_m, b_{m+1}, \ldots, b_{n-1}) \\ &\quad = \tilde{g}_n(x; b_0, \ldots, b_{m-1}, b_m, b_{m+1}, \ldots, b_{n-1}) \\ &\qquad - \binom{n}{m} \tilde{g}_{n-m}(b_m+\delta_m; b_m, b_{m+1}, \ldots, b_{n-1}) \tilde{g}_m(x; b_0, \ldots, b_{m-1}). \end{aligned} \tag{3.11}$$

Applying the perturbation formula repeatly, we get

$$\begin{aligned} &\tilde{g}_n(x; b_0+\delta_0, b_1+\delta_1, \ldots, b_{n-1}+\delta_{n-1}) \\ &\quad = \tilde{g}_n(x; b_0, \ldots, b_{n-1}) \\ &\qquad - \sum_{i=0}^{n} \binom{n}{i} \tilde{g}_{n-i}(b_i+\delta_i; b_i, \ldots, b_{n-1}) \tilde{g}_i(x; b_0+\delta_0, \ldots, b_{i-1}+\delta_{i-1}). \end{aligned} \tag{3.12}$$

9. Binomial expansion. If we expand $\tilde{g}_n(x+y; b_0, \ldots, b_{n-1})$ using the basis $\{x^{(n)}\}$, we can get

$$\tilde{g}_n(x+y; b_0, \ldots, b_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} \tilde{g}_{n-i}(y; b_i, \ldots, b_{n-1}) x^{(i)}. \tag{3.13}$$

To see this, note that $\Delta(x+y)^{(i)} = i(x+y)^{(i-1)}$, and $\Delta\tilde{g}_n(x+y; b_0, \ldots, b_{n-1}) = n\tilde{g}_{n-1}(x+y; b_0, \ldots, b_{n-1})$. Now apply Δ to both side of Equation (3.13) and set $x = 0$. Equation (3.13) follows from induction.

Difference Goncarov polynomial of low degrees can be easily computed by the determinant formula or the summation formula. For example,

$$\begin{aligned} \tilde{g}_0(y) &= 1, \\ \tilde{g}_1(y; b_0) &= y^{(1)} - b_0^{(1)}, \\ \tilde{g}_2(y; b_0, b_1) &= y^{(2)} - 2b_1^{(1)} y^{(1)} + 2b_0^{(1)} b_1^{(1)} - b_0^{(2)}, \\ \tilde{g}_3(y; b_0, b_1, b_2) &= y^{(3)} - 3b_2^{(1)} y^{(2)} + (6b_1^{(1)} b_2^{(1)} - 3b_1^{(2)}) y^{(1)} \\ &\quad - b_0^{(3)} + 3b_0^{(2)} b_2^{(1)} - 6b_0^{(1)} b_1^{(1)} b_2^{(1)} + 3b_0^{(1)} b_1^{(2)}. \end{aligned}$$

In the following special cases, difference Goncarov polynomials have a nice closed-form expression.

Case 1 $b_i = b$ for all i. Then

$$\tilde{g}_n(x, b, \ldots, b) = (x-b)^{(n)}.$$

Case 2 $b_i = y + (i-1)b$ forms an arithmetic progression. Then we have the difference analog of Abel polynomials:

$$\tilde{g}_n(x, y, y+b, \ldots, y+(n-1)b) = \begin{cases} (x-y)(x-y-nb+1)^{(n-1)} & n > 0; \\ 1 & n = 0. \end{cases}$$

To see this, verify the difference relation that $\Delta \tilde{g}_n(x; b_0, \ldots, b_{n-1}) = n\tilde{g}_{n-1}(x; b_1, \ldots, b_{n-1})$ and $\tilde{g}_n(b_0; b_0, \ldots, b_{n-1}) = \delta_{0n}$. In particular, $\tilde{g}_n(0; -1, \ldots, -n) = \frac{n!}{n+1}\binom{2n}{n} = n!C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the famous Catalan number.

4 Difference Goncarov Polynomials and Lattice Paths

In this section we describe a combinatorial decomposition which allows us to relate the difference Goncarov polynomials with certain lattice paths in plane.

Let x, n be positive integers. Consider lattices paths from $(0, 0)$ to $(x-1, n)$ with steps $(1, 0)$ or $(0, 1)$. Denote by the sequence $(x_0, \ldots, x_n)$ such a path whose right-most point on the i-th row is (x_i, i). Obviously, we always have $x_n = x-1$. Given $b_0 \le b_1 \le \cdots \le b_{n-1} \le x$, let $LP_n(b_0, \ldots, b_{n-1})$ be the number of paths $(x_0, \ldots, x_{n-1})$ from $(0, 0)$ to $(x-1, n)$ with steps $(1, 0)$ and $(0, 1)$ such that $x_i < b_i$ for $0 \le i \le n-1$.

It is well-known that the total number of the paths from $(0, 0)$ to $(x-1, n)$ in the grid $(x-1) \times n$ is

$$LP_n(x, \ldots, x) = \binom{x+n-1}{n} = \frac{x^{(n)}}{n!}.$$

Another way of counting paths in the grid $(x-1) \times n$ is to decompose the paths into several classes as follows. Let $(x_0, \ldots, x_n)$ be such a path and i be the first row that $x_i \ge b_i$. Each of such paths consists of three parts: the first part is a path from $(0, 0)$ to $(b_i - 1, i)$ that never touches the points (b_j, j) for $j = 0, 1, \ldots, i$, the second path consists of one step $(1, 0)$, from $(b_i - 1, i)$ to (b_i, i), and the third part is a path that goes from (b_i, i) to $(x-1, n)$. The number of paths of the first kind is $LP_i(b_0, \ldots, b_{i-1})$, while that of the third kind is $\binom{x-1-b_i+n-i}{n-i} = \frac{(x-b_i)^{(n-i)}}{(n-i)!}$. Therefore the total number of paths is

$$\sum_{i=0}^{n} LP_i(b_0, \ldots, b_{i-1}) \frac{(x-b_i)^{(n-i)}}{(n-i)!}.$$

So

$$x^{(n)} = \sum_{i=0}^{n} n! LP_i(b_0, \ldots, b_{i-1}) \frac{(x-a_i)^{(n-i)}}{(n-i)!}. \tag{4.1}$$

Comparing Equations (3.4) and (4.1), we get

Theorem 4.1

$$LP_i(b_0, \ldots, b_{i-1}) = \frac{1}{i!} \tilde{g}_i(x; x-b_0, \ldots, x-b_{n-1}).$$

In particular,

$$LP_n(b_0, \ldots, b_{n-1}) = \frac{1}{n!} \tilde{g}_n(0; -b_0, \ldots, -b_{n-1}). \tag{4.2}$$

Using the identity $(-x)^{(n)} = (-1)^n x(x-1)(x-2)\cdots(x-n+1) = (-1)^n x_{(n)}$ where $x_{(n)}$ is the lower factorial, and the determinant formula for $\tilde{g}_n$, we get

$$LP_n(b_0, \ldots, b_{n-1}) = \det \left[\binom{b_i}{j-i+1} \right]_{0 \le i,j \le n-1}.$$

An equivalent description for $LP_n(b_0, \ldots, b_{n-1})$ is the number of integer points in certain n-dimensional polytope considered by Pitman and Stanley in [8]. Let

$$\Pi_n(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : y_i \ge 0 \text{ and } y_1 + y_2 + \cdots + y_i \le x_1 + \cdots + x_i \text{ for all } 1 \le i \le n\},$$

Pitman and Stanley computed the number of integer points in the polytope Π_n when $x_1, \ldots, x_n$ are positive integers, and gave the formula

$$N(\Pi_n(\mathbf{x})) = \sum_{\mathbf{k} \in K_n} \frac{(x_1+1)^{(k_1)}}{k_1!} \prod_{i=2}^{n} \frac{x_i^{(k_i)}}{k_2!},$$

where

$$K_n = \{\mathbf{k} \in \mathbb{N}^n : \sum_{i=1}^{i} k_i \ge j \text{ for all } 1 \le i \le n-1 \text{ and } \sum_{i=1}^{n} k_i = n\}.$$

Letting $b_0 = x_0 + 1$, $b_i = 1 + \sum_{j=0}^{i} x_i$. Every integer point $\mathbf{y} = (y_1, \ldots, y_n) \in \Pi_n(\mathbf{x})$ corresponds uniquely to a lattice path $0 \le r_0 \le r_1 \le \cdots \le r_{n-1}$ where $r_i = \sum_{i=0}^{i} y_i < b_i$ for all i. Hence

$$N(\Pi_n(\mathbf{x})) = LP_n(b_0, b_1, \ldots, b_{n-1}) = \det \left[\binom{b_i}{j-i+1} \right]_{0 \le i,j \le n-1}. \tag{4.3}$$

Formula (4.3) can also be derived from the Steck formula (c.f. Theorem 7.3) on the number of lattice paths lying between two given increasing sequences [11, 12]. The detailed can be found in the monograph [4] and corresponding theory of biorthogonal polynomials are presented in Section 7.

As an application of (4.2), let $b_i = i$. Then $LP_n(1,2,\ldots,n)$ counts the number of Dyck paths. We have $LP_n(1,\ldots,n) = \frac{1}{n!}\tilde{g}_n(0;-1,\ldots,-n) = \frac{1}{n+1}\binom{2n}{n}$, again obtain the famous Catalan number. In general, we can consider the number of lattice paths from $(0,0)$ to $(r+\mu n, n)$ $(r,\mu \in \mathbb{P})$, which never touch the line $x = r + \mu y$. This is just the case where $b_i = r + (i-1)\mu$, and the number is given by

$$\begin{aligned} LP_n(r, r+\mu, r+2\mu, \ldots, r+(n-1)\mu) &= \frac{1}{n!}\tilde{g}_n(0; -r, -r-\mu, \ldots, -r-(n-1)\mu) \\ &= \frac{1}{n!} r(r+n\mu+1)^{(n-1)} \\ &= \frac{r}{r+n(\mu+1)}\binom{r+n(\mu+1)}{n}, \end{aligned} \tag{4.4}$$

a well-known result. (See, for example, [4, p.9]. In particular, for $r = 1$ and $\mu = k$, it counts the number of lattice paths from the origin to (kn, n) that never pass below the line $y = x/k$. The formula (4.4) becomes $\frac{1}{kn+1}\binom{(k+1)n}{n}$, the nth k-Catalan number [10, p. 175].

We can reinterpret the perturbation formula (3.12) using paths. Given two paths $(a_0,\ldots,a_{n-1})$ and $(c_0, c_1, \ldots, c_{n-1})$ with $a_i \le c_i$, We consider all paths that never touch $(c_0, c_1, \ldots, c_{n-1})$. First, it is $LP_n(c_0,\ldots,c_{n-1})$ as defined. Secondly, we can also count the paths that never touch $(c_0,\ldots,c_{n-1})$, while they touch the path $(a_0,\ldots,a_n)$ on i-th row for the first time. The total number of such paths is $LP_i(a_0,\ldots,a_{i-1})LP_{n-i}(c_i-a_i, c_{i+1}-a_i,\ldots,c_{n-1}-a_i)$. So we have the formula

$$\begin{aligned} LP_n(c_0,\ldots,c_{n-1}) &= \sum_{i=0}^{n} LP_i(a_0,\ldots,a_{i-1})LP_{n-i}(c_i-a_i, c_{i+1}-a_i, \ldots, c_{n-1}-a_i) \\ &\quad + LP_n(a_0,\ldots,a_n). \end{aligned} \tag{4.5}$$

Converting the equation using difference Goncarov polynomials, we have

$$\begin{aligned} &\tilde{g}_n(x; x-a_0, x-a_1, \ldots, x-a_{n-1}) \\ &\quad = \tilde{g}_n(x; x-c_0, \ldots, x-c_{n-1}) \\ &\qquad - \sum_{i=0}^{n}\binom{n}{i}\tilde{g}_{n-i}(0; a_i-c_i, \ldots, a_i-c_{n-1})\tilde{g}_i(x; x-a_0, \ldots, x-a_{i-1}). \end{aligned}$$

Replacing $x - c_i$ with b_i, $c_i - a_i$ with δ_i, and using the shift formula (3.10) on

$\tilde{g}_{n-i}(0; a_i - c_i, \ldots, a_i - c_{n-1})$ by

$$\begin{aligned}\tilde{g}_{n-i}(0; a_i - c_i, \ldots, a_i - c_{n-1}) &= \tilde{g}_{n-i}(0; -\delta_i, b_{i+1} - b_i - \delta_i, \ldots, b_{n-1} - b_i - \delta_i) \\ &= \tilde{g}_{n-i}(b_i + \delta_i; b_i, b_{i+1}, \ldots, b_{n-1}),\end{aligned}$$

we get the perturbation formula (3.12) again.

REMARK. Let $b_0 \leq b_1 \leq b_{n-1}$ be a sequence of integers. Denote by $\mathbf{LP}^{<}_{\mathbf{n}}$ the set of integer sequences $(r_0 < r_1 < \cdots < r_{n-1})$ such that $0 \leq r_i < b_i$ for $i = 0, 1, \ldots, n-1$. Then $LP^{<}_n$, the cardinality of $\mathbf{LP}^{<}_{\mathbf{n}}$, can be obtained as follows: Let $s_i = r_i - (i-1)$. Then $s_0 \leq s_1 \leq \cdots \leq s_{n-1}$ and $0 \leq s_i < b_i - (i-1)$. Hence $LP^{<}_n(b_0, \ldots, b_{n-1}) = LP_n(b_0, b_1 - 1, \ldots, b_{n-1} - n - 1)$.

Alternatively, we can use the forward difference operator Δ_f and its basic polynomials $x_{(n)} = x(x-1)\ldots(x-n+1)$ to replace Δ and $x^{(n)}$ in (3.1), where $\Delta_f p(x) = p(x+1) - p(x)$. Explicitly, let $\psi_S(\Delta_f) = \sum_{r=0}^{\infty} \frac{(b_s)_{(r)}}{r!} \Delta_f^{r+s}$. Denote the corresponding sequence of biorthogonal polynomials by $\tilde{g}_{f,n}(x; b_0, \ldots, b_{n-1})$. The determinant formula of $\tilde{g}_{f,n}(x; b_0, \ldots, b_{n-1})$ is obtained from (3.2) by replacing each upper factorial $a^{(i)}$ with the lower factorial $a_{(i)} = a(a-1)\ldots(a-i+1)$. Under this setting, we have $LP^{<}_n(b_0, \ldots, b_{n-1}) = \frac{1}{i!}\tilde{g}_{f,n}(0; -b_0, \ldots, -b_{n-1})$.

The above two approaches yield the following determinant formulas for $LP^{<}_n(b_0, \ldots, b_{n-1})$.

$$LP^{<}_n(b_0, \ldots, b_{n-1}) = \det\left[\binom{b_i - i}{j - i + 1}\right]_{0 \leq i,j < n} = \det\left[\binom{b_i + j - i}{j - i + 1}\right]_{0 \leq i,j < n}.$$

5 q-Goncarov Polynomial

For $\mathbf{u}$-parking functions, the sum enumerator $S_n(q, \mathbf{u}) = \sum_{(a_1, \ldots, a_n)} q^{a_1 + a_2 + \cdots + a_n}$, where $(a_1, \ldots, a_n)$ ranges over all $\mathbf{u}$-parking functions, is just the specialization of the (differential) Goncarov polynomials where u_i is replaced with $1 + q + \cdots + q^{u_i - 1}$. This is not the case for lattice paths and difference Goncarov polynomials. Define the *area*-enumerator of lattice paths to be

$$\mathrm{Area}_n(q; \mathbf{b}) := \sum_{(x_0, \ldots, x_{n-1}) \in \mathbf{LP}_n(\mathbf{b})} q^{x_0 + x_1 + \cdots + x_{n-1}}, \tag{5.1}$$

where $\mathbf{LP}_n(\mathbf{b})$ is the set of lattice paths from $(0,0)$ to $(x-1, n)$ $(x - 1 \geq b_{n-1})$ that never touch $(b_0, b_1, \ldots, b_{n-1})$. Note that $x_0 + x_1 + \cdots + x_{n-1}$ is the area of the region bounded by the path and the lines $x = 0$ and $y = n$. To study $\mathrm{Area}_n(q; \mathbf{b})$, we develop the q-analog of difference Goncarov polynomials.

We use the following the conventions that $(n)_q = \frac{1-q^n}{1-q}$; $(n)_q! = (1)_q \cdots (n)_q$; and the rising q-factorial

$$(A; q)_n = \begin{cases} (1-A)\cdots(1-Aq^{n-1}) & \text{if} \quad n > 0, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

Let $p(y)$ be a polynomial in the ring $F(q)[y]$. Define

$$\Delta_q p(y) = \frac{p(y) - p(y/q)}{(1-q)y/q}.$$

It is easy to check that $\Delta_q p(y)$ is a polynomial of y whose degree is one less than that of $p(y)$.

Observe that the polynomials $p_n(y) = (y;q)_n$ form a basis of the ring $F(q)[y]$; $\Delta_q p_n(y) = (n)_q p_{n-1}(y)$; and $\Delta_q^i p_n(y)\big|_{y=1} = 0$, whenever $i < n$. Let $\psi_{q,s}(\Delta_q)$, $s = 0, 1, 2, \ldots$, be the sequence of linear operators

$$\psi_{q,s}(D_q) = \sum_{s=0}^{\infty} \frac{(b_s;q)_r}{(r)_q!} \Delta_q^{r+s}, \tag{5.2}$$

and define the difference q-Goncarov polynomials $g_n(q;y;\mathbf{b}) = g_n(q;y;b_0,\ldots,b_{n-1})$ to be the sequence of polynomials biorthogonal to $\psi_{q,s}(\Delta_q)$, i.e.,

$$\psi_{q,s}(\Delta_q) g_n(y;\mathbf{b};q)\big|_{y=1} = (n)_q! \delta_{sn}.$$

Similar properties satisfied by the regular Goncarov polynomials can be generalized to a q-analog for the difference q-Goncarov polynomials. We list the main results in the following.

1. Determinantal formula.

$$g_n(q;y;b_0,\ldots,b_{n-1}) = (n)_q! \begin{vmatrix} 1 & (b_0;q)_1 & \frac{(b_0;q)_2}{(2)_q!} & \frac{(b_0;q)_3}{(3)_q!} & \cdots & \frac{(b_0;q)_{n-1}}{(n-1)_q!} & \frac{(b_0;q)_n}{(n)_q!} \\ 0 & 1 & b_1 & \frac{(b_1;q)_2}{(2)_q!} & \cdots & \frac{(b_1;q)_{n-2}}{(n-2)_q!} & \frac{(b_1;q)_{n-1}}{(n-1)_q!} \\ 0 & 0 & 1 & b_2 & \cdots & \frac{(b_2;q)_{n-3}}{(n-3)_q!} & \frac{(b_2;q)_{n-1}}{(n-2)_q!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & b_{n-1} \\ 1 & (y;q)_1 & \frac{(y;q)_2}{(2)_q!} & \frac{(y;q)_3}{(3)_q!} & \cdots & \frac{(y;q)_{n-1}}{(n-1)_q!} & \frac{(y;q)_n}{(n)_q!} \end{vmatrix}.$$

2. Expansion formula. For any polynomial $p(y) \in F(q)[y]$,

$$p(y) = \sum_{i=0}^{n} \frac{\psi_{q,i}(\Delta_q)(p(y))\big|_{y=1}}{(i)_q!} g_n(q;y;b_0,\ldots,b_{i-1}), \tag{5.3}$$

   To verify, apply D_i on both sides and then set $y = 1$.

3. Linear recursion.

$$(y;q)_n = \sum_{i=0}^{n} \binom{n}{i}_q (b_i;q)_{n-i} g_i(q;y;b_0,\ldots,b_{i-1}). \tag{5.4}$$

4. Appell Relation. Since
$$\sum_n \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty},$$
where $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, we have the generating function
$$(yt;q)_\infty = \sum_{i=0}^{\infty} g_i(q;y;b_0,\ldots,b_{i-1}) \frac{(b_i t;q)_\infty}{(i)_q!} t^i$$

5. Difference relation.
$$\Delta_q g_n(q;y;b_0,\ldots,b_{n-1}) = (n)_q g_{n-1}(q;y;b_1,\ldots,b_{n-1}), \tag{5.5}$$
with the initial conditions
$$g_n(q;b_0;b_0,\ldots,b_{n-1}) = \delta_{0n}. \tag{5.6}$$

6. Binomial expansion. The binomial expansion of Goncarov polynomials becomes
$$g_n(q;ty;b_0,\ldots,b_{n-1};q) = \sum_{i=0}^{n} \binom{n}{i}_q t^i g_{n-i}(q;t;b_i,\ldots,b_{n-1})(y;q)_i. \tag{5.7}$$
This is because $\Delta_q (ty;q)_i = (i)_q t(ty;q)_{i-1}$, and
$$\Delta_q g_n(q;ty;b_0,\ldots,b_{n-1}) = (n)_q t g_{n-1}(q;ty;b_1,\ldots,b_{n-1}).$$
Now apply Δ_q to both side of the equation and set $y=1$.

7. Summation formula. Let $y=q^x$ and $b_i = q^{a_i}$, where x and a_i are integers, we have
$$g_n(q;y;b_0,\ldots,b_{n-1}) = (1-q)\sum_{i=a_0+1}^{x} q^{i-1}(n)_q g_{n-1}(q;q^i;b_1,\ldots,b_{n-1}),$$
where the sum is defined the same as in 3.9. This is because
$$g_n(q;y;b_0,\ldots,b_{n-1}) = g_n(q;q^{x-1};b_0,\ldots,b_{n-1}) + (1-q)q^{x-1}(n)_q g_{n-1}(q;q^x;b_1,\ldots,b_{n-1})$$
and the initial condition (5.6). Iterate it we obtained the *sum formula*
$$g_n(q;y;b_0,\ldots,b_{n-1}) = (1-q)^n (n)_q! \sum_{i_1=a_0+1}^{x} q^{i_1-1}\left(\sum_{i_2=a_1+1}^{i_1} q^{i_2-1}\left(\cdots \sum_{i_n=a_{n-1}+1}^{i_{n-1}} q^{i_n-1}\right)\right) \tag{5.8}$$
From this we derive the *shift formula*:
$$g_n(q;yq^\xi;b_0q^\xi,\ldots,b_{n-1}q^\xi) = q^{n\xi} g_n(q;y;b_0,\ldots,b_{n-1}). \tag{5.9}$$
Since $g_n(q;y;\mathbf{b})$ is a polynomial of y and b's over a field and the equation above holds for infinitely many y's and b's, it holds for all y and b's.

Examples of the q-Goncarov polynomials follow.

$$\begin{aligned}
g_0(q;y) &= 1,\\
g_1(q;y;b_0) &= (y;q)_1-(b_0;q)_1,\\
g_2(q;y;b_0,b_1) &= (y;q)_2-(2)_q!(b_1;q)_1(y;q)_1+(2)_q!(b_0;q)_1(b_1;q)_1-(b_0;q)_2,\\
g_3(q;y;b_0,b_1,b_2) &= (y;q)_3-(3)_q(b_2;q)_1(y;q)_2+((3)_q!(b_1;q)_1(b_2;q)_1-(3)_q(b_1;q)_2)(y;q)_1\\
&\quad -(b_0;q)_3+(3)_q(b_0;q)_2(b_2;q)_1-(3)_q!(b_0;q)_1(b_1;q)_1(b_2;q)_1+(3)_q(b_0;q)_1(b_1;q)_2.
\end{aligned}$$

In particular, in some special cases we have nice closed formula,

$$g_n(q;q^x;q^b,\ldots,q^b)=q^{nb}(q^{x-b};q)_n$$

and

$$g_n(q;q^x;q^y,q^{y+1},\ldots,q^{y+n-1})=(-1)^n q^{\binom{n}{2}-nx}(q^{y-x};q)_n.$$

6 q-Goncarov Polynomials and Area of Lattice Paths

Define the q-upper factorial $x_q^{(n)}=(1-q^x)\cdots(1-q^{x+n-1})/(1-q)^n$. In this section we use the difference q-Goncarov polynomial developed in the previous section to represent the area-enumerator of lattice paths with upper constraint.

Consider an $(x-1)$ by n grid consisting of vertical and horizontal lines. Let $(x_0,\ldots,x_{n-1})$ be a path that goes from $(0,0)$ to $(x-1,n)$ along the grid, while the right-most point on the i-th row is (x_i,i) for $i=0,1,\ldots,n-1$. Given a path $(b_0,b_1,\ldots,b_{n-1})$ with $b_0\le b_1\le\cdots\le b_{n-1}\le x-1$, let $\mathrm{Area}_n(q;\mathbf{b})$ be given in Formula (5.1). We establish a recurrence of $\mathrm{Area}_n(q;\mathbf{b})$ by computing the area-enumerator of $\mathrm{Area}_n(q;x-1,x-1,\ldots,x-1)$ in two ways.

First, it is well-known that the area-enumerators of all the paths in the grid $(x-1)\times n$ is

$$\mathrm{Area}_n(q;x-1,\ldots,x-1)=\binom{x+n-1}{n}_q=\frac{x_q^{(n)}}{(n)_q!}.$$

Now apply the decomposition as in §4. Let $(x_0,\ldots,x_n)$ be a path in $(x-1)\times n$ for which i be the first row that the path touches the path $(b_0,\ldots,b_{n-1})$, i.e., $x_i\ge b_i$. Each of such paths consists of two parts: the first part is a path from $(0,0)$ to (b_i-1,i) that never touches the path $(b_0,\ldots,b_{i-1})$, the second part consists of one horizontal step from (b_i-1,i) to (b_i,i), and the third part is a path that goes from (b_i,i) to $(x-1,n)$. The area contributed by the first part is $\mathrm{Area}_i(q;b_0,\ldots,b_{i-1})$, while that of the second kind and the third is $q^{b_i(n-i)}\binom{x-1-b_i+n-i}{n-i}_q = q^{b_i(n-i)}\frac{(x-b_i)_q^{(n-i)}}{(n-i)_q!}$. Therefore the area-enumerator of all the paths is

$$\sum_{i=0}^{n} \text{Area}_i(q;b_0,\ldots,b_i)q^{b_i(n-i)}\frac{(x-b_i)_q^{(n-i)}}{(n-i)_q!}.$$

So

$$x_q^{(n)} = \sum_{i=0}^{n} \frac{(n)_q!}{(n-i)_q!}\text{Area}_i(q;b_0,\ldots,b_i)q^{(n-i)b_i}(x-b_i)_q^{(n-i)}, \tag{6.1}$$

which leads to

$$(q^x;q)_n = \sum_{i=0}^{n} \frac{(n)_q!}{(n-i)_q!}(1-q)^i\left[(q^{x-b_i};q)_{n-i}\right]q^{(n-i)b_i}\text{Area}_i(q;b_0,\ldots,b_i). \tag{6.2}$$

Comparing equations (5.4) and (6.2), we need to make the power of q on the right-hand side of equation (6.2) depending only on i and b_i. To achieve this, we replace q by $\frac{1}{q}$ in equation (6.2). Then $(q^x;q)_n$ becomes $(-1)^n q^{-(nx+\binom{n}{2})}(q^x;q)_n$, and $(n)_q!$ becomes $q^{-\binom{n}{2}}(n)_q!$. Substituting into (6.2) and reorganizing the equation, we get

$$(q^x;q)_n = \sum_{i=0}^{n} \binom{n}{i}_q (i)_q!(-1)^i\left[(q^{x-b_i};q)_{n-i}\right]q^{ix}(1-\frac{1}{q})^i\text{Area}_i(\frac{1}{q};b_0,\ldots,b_i). \tag{6.3}$$

Comparing equations (5.4) and (6.3), we obtain

$$\text{Area}_i(\frac{1}{q};b_0,\ldots,b_i) = \frac{(-1)^i}{(i)_q!q^{ix}(1-\frac{1}{q})^i}g_n(q;q^x;q^{x-b_0},\ldots,q^{x-b_{i-1}}).$$

Let $f(q;x,b_0,\ldots,b_{n-1})$ be a polynomials of q with parameters $x,b_0,\ldots,b_{n-1}$ given by

$$f(q;x,b_0,\ldots,b_{n-1}) = g_n(q;q^x;q^{x-b_0},\ldots,q^{x-b_{i-1}}).$$

Then

Theorem 6.1 *The area-enumerators of lattice paths in the rectangle $(x-1)\times n$ that stays strictly above the path $(b_0,\ldots,b_n)$ is*

$$\begin{aligned}\text{Area}_n(q;\mathbf{b}) &= \frac{(-1)^n}{(1-q)^n(n)_q!}q^{\frac{n(n-1)}{2}+nx}f(\frac{1}{q};x,b_0,\ldots,b_{n-1}) \\ &= \frac{(-1)^n}{(1-q)^n(n)_q!}q^{\frac{n(n-1)}{2}}f_n(\frac{1}{q};0,b_0,\ldots,b_{n-1}). \end{aligned} \tag{6.4}$$

7 Two-Boundary Extensions

Goncarov polynomials can be extended to represent parking functions and lattice paths with both both upper and lower boundaries.

7.1 Parking functions with two-sided boundary

u-parking functions are integer sequences whose order statistics is bounded by a prefixed sequence **u** from above. We may consider parking functions with both upper and lower constraints. More precisely, let $r_1 \leq r_2 \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_n$ be two sequence of non-decreasing integers. A $(\mathbf{r}, \mathbf{s})$-parking function of length n is a sequence $(x_1, \ldots, x_n)$ whose order statistics satisfy $r_i \leq x_{(i)} < s_i$. Denoted by $P_n(\mathbf{r}, \mathbf{s}) = P_n(r_1, \ldots, r_n; s_1, \ldots, s_n)$ the number of $(\mathbf{r}, \mathbf{s})$-parking functions of length n. The formula $P_n(\mathbf{r}, \mathbf{s})$ can also be expressed as biorthogonal polynomials.

Let $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ be two sequences of numbers. Define the *extended Goncarov polynomials*

$$g_n^\dagger(x; \mathbf{a}, \mathbf{b}) = g_n^\dagger(x; a_0, a_2, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1}), n = 0, 1, 2, \ldots,$$

to be the sequence of polynomials biorthogonal to the operators

$$\varphi_s(D) = D^s \sum_{r=0}^{\infty} \frac{(b_s - a_{s+r-1})_+^r D^r}{r!}, \tag{7.1}$$

where $x_+ = \max(x, 0)$. (Here we set $a_{-1} = 0$.) By the determinant formula (2.3),

$$g_n^\dagger(x; \mathbf{a}, \mathbf{b}) = n! \begin{vmatrix} 1 & (b_0 - a_0)_+ & \frac{(b_0-a_1)_+^2}{2!} & \frac{(b_0-a_2)_+^3}{3!} & \cdots & \frac{(b_0-a_{n-2})_+^{n-1}}{(n-1)!} & \frac{(b_0-a_{n-1})_+^n}{n!} \\ 0 & 1 & (b_1 - a_1)_+ & \frac{(b_1-a_2)_+^2}{2!} & \cdots & \frac{(b_1-a_{n-2})_+^{n-2}}{(n-2)!} & \frac{(b_1-a_{n-1})_+^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & (b_2 - a_2)_+ & \cdots & \frac{(b_2-a_{n-2})_+^{n-3}}{(n-3)!} & \frac{(b_2-a_{n-1})_+^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & (b_{n-1} - a_{n-1})_+ \\ 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \end{vmatrix}.$$

In particular, $g_n^\dagger(0; \mathbf{a}, \mathbf{b})) = (-1)^n n! \det[(b_i - a_j)_+^{j-i+1}/(j-i+1)!]$. By the linear recurrence equation (2.5), we have

$$x^n = \sum_{i=0}^{n} \binom{n}{i} (b_i - a_{n-1})_+^{n-i} g_i^\dagger(x; \mathbf{a}, \mathbf{b}).$$

It follows that for $n \geq 1$,

$$\sum_{i=0}^{n} \binom{n}{i} (b_i - a_{n-1})_+^{n-i} g_i^\dagger(0; \mathbf{a}, \mathbf{b}) = 0. \tag{7.2}$$

The sequence $\{g_n^\dagger(0, \mathbf{a}, \mathbf{b})$ is uniquely determined by the above recurrence and initial values $g_0^\dagger(0; \mathbf{a}, \mathbf{b}) = 1$, $g_1^\dagger(0; \mathbf{a}, \mathbf{b}) = -(b_0 - a_0)_+$.

Theorem 7.1

$$P_n(r_1, \ldots, r_n; s_1, \ldots, s_n) = (-1)^n g_n^\dagger(0; r_1, \ldots, r_n; s_1, \ldots, s_n). \tag{7.3}$$

To prove Theorem 7.1, it is sufficient to show that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (s_{i+1} - r_n)_+^{n-i} P_i(\mathbf{r}, \mathbf{s}) = 0, \tag{7.4}$$

for $n \geq 0$, and $P_1(\mathbf{r}, \mathbf{s}) = (s_1 - r_1)_+$. The initial value is clear. In the following we give two proof of equation (7.4). The first one is based on a weighted version of inclusion-exclusion principle. The second is an involution on the set of "marked" parking functions, which reveals some intrinsic structures of two-sided parking functions.

First Proof of (7.4). Let $M(S)$ be the set of all sequences α of length n such that $\alpha|_S$ is a $(\mathbf{r}, \mathbf{s})$-parking function of length $|S|$, and each term in $\alpha|_{S^c}$ lies in $[r_n, s_{i+1})$, where $S^c = [n] \setminus S$. Then (7.4) is equivalent to $\sum_S (-1)^{|S|} |M(S)| = 0$, where the sum ranges over all subsets $S \subseteq [n]$. For any sequence α, let $T[\alpha] = \{S : \alpha \in M(S)\}$. It is sufficient to show that

$$\sum_{T \in T[\alpha]} (-1)^{|T|} = 0. \tag{7.5}$$

Observe that if $\alpha \in M(S)$ and $S \subset S'$, then $\alpha \in M(S')$. Hence $T[\alpha]$ is a filter in the power set of $[n]$. $T[\alpha] \neq \emptyset$ if and only if α is a $(\mathbf{r}, \mathbf{s})$-parking function. When $T[\alpha] \neq \emptyset$, let $S_1, \ldots, S_r$ be the minimal elements of $T[\alpha]$. $S_1, \ldots, S_r$ satisfy the following properties.

1. $|S_i| < n$. For any $(\mathbf{r}, \mathbf{s})$-parking function α, deleting the largest element which is in $[r_n, s_n)$, the remaining is a $(\mathbf{r}, \mathbf{s})$-parking function of length $n-1$. Hence $T[\alpha] \neq \{[n]\}$.
2. $|S_1| = |S_2| = \cdots = |S_r| = k$ for some $k < n$. Assume $k = |S_1| < |S_2| = \ell$. The condition $\alpha \in M(S_1)$ implies all terms of α are less than s_{k+1}, and at least $n-k$ of them are larger than or equal to r_n. In particular, the largest element in $\alpha|_{S_2}$ lies in $[r_n, s_{k+1})$. Then S_2 is not minimal.

3. $S_1 \cup S_2 \cup \cdots \cup S_r \neq [n]$. Otherwise, every term in α appears in some $(\mathbf{r}, \mathbf{s})$-parking function of length $k < n$, and hence less than s_k. And any term in a position of $S_1 \setminus S_2$ is greater than or equal to r_n. Hence the minimal element of $T[\alpha]$ has length $\leq |S_1| - 1$, a contradiction.

Denoted by $\mathcal{F}(S_1, \ldots, S_r)$ the filter of the power set of $[n]$ generated by $S_1, \ldots, S_r$, and

$$W(\mathcal{F}(S_1, \ldots, S_r)) = \sum_{T \in \mathcal{F}(S_1, \ldots, S_r)} w(T),$$

for a weight function $w(T)$. Note that $\mathcal{F}(S_1, \ldots S_r) = \mathcal{F}(S_1) \cup \ldots \mathcal{F}(S_r)$, and $\mathcal{F}(S_i) \cap \mathcal{F}(S_j) = \mathcal{F}(S_i \cup S_j)$, using Inclusion-Exclusion, we have

$$W(\mathcal{F}(S_1, \ldots, S_r)) = \sum_i W(\mathcal{F}(S_i)) - \sum_{i<j} W(\mathcal{F}(S_i \cup S_j)) + \sum_{i<j<k} W(\mathcal{F}(S_i \cup S_j \cup S_k)) - \cdots .$$

Letting the weight $w(T) = (-1)^{|T|}$. Formula (7.5) follows from the equation $W(\mathcal{F}(S)) = (-1)^{n-|S|} = 0$ whenever $S \neq [n]$. □

Second Proof of (7.4). We give a bijective proof for the equivalent form

$$\sum_{i \text{ even}} \binom{n}{i} (s_{i+1} - r_n)_+^{n-i} P_i(\mathbf{r}, \mathbf{s}) = \sum_{i \text{ odd}} \binom{n}{i} (s_{i+1} - r_n)_+^{n-i} P_i(\mathbf{r}, \mathbf{s}). \tag{7.6}$$

The left-hand side of (7.6) is the cardinality of the set M of pairs (α, S) where α is a sequence of length n, $S \subset [n]$ with $|S|$ even, such that $\alpha|_S$ is a $(\mathbf{r}, \mathbf{s})$-parking function of length $|S|$, and any term in $\alpha|_{S^c}$ lies in $[r_n, s_{|S|+1})$. The right-hand side of (7.6) is the cardinality of the set N of pairs (α, S) where (α, S) is similar as those appeared in M, except that $|S|$ being odd.

For a sequence α, let $m = \max(\alpha)$ be the first maximal entry of α. Let $pos(m)$ be the position of m. Define $\sigma : M \mapsto N$ by letting $\sigma(\alpha, S) = (\alpha, T)$ where

$$T = \begin{cases} (\alpha, S \setminus \{pos(m)\}), & \text{if } pos(m) \in S, \\ (\alpha, S \cup \{pos(m)\}), & \text{if } pos(m) \notin S. \end{cases}$$

The map σ is well-defined: For any pair (α, S) with $|S|$ even, clearly $|T|$ is odd.
Case 1. If $pos(m) \in S$, then deleting m from the subsequence in S, we obtain a (r, s)-parking function of length $|S| - 1 = |T|$. The condition that $m = \max(\alpha)$ and $m \in [r_{|S|}, s_{|S|})$ implies that for any term x in $\alpha|_{T^c}$, $x \leq m < s_{|S|} = s_{|T|+1}$. In addition, if $S^c \neq \emptyset$, then $m \geq x \geq r_n$ for any $x \in S^c$; if $S^c = \emptyset$, then α itself is a (r, s)-parking function of length n, hence $m \geq r_n$. This proves that in the case $pos(m) \in S$, $(\alpha, T) \in N$.
Case 2. If $pos(m) \notin S$, then any term $x \in S^c$ lies in $[r_n, s_{|S|+1}) \subseteq [r_n, s_{|S|+2})$. As $m \in [r_n, s_{|S|+1})$, joining m to the subsequence on S will result in a (r, s)-parking function of length $|S| + 1 = |T|$. In both cases, σ maps a pair in M to a pair in N.

It is easily seen that σ has the inverse map $\sigma^{-1}(\alpha, T) = (\alpha, S)$ where

$$S = \begin{cases} (\alpha, T \setminus \{pos(m)\}), & \text{if } pos(m) \in T, \\ (\alpha, T \cup \{pos(m)\}), & \text{if } pos(m) \notin T. \end{cases}$$

This proved that σ is a bijection from M to N. □

Equation (7.1) should be compared with following formula of Steck [11, 12] for the cumulative distribution function of the random vector of order statistics of n independent random vaiables with uniform distribution on an interval. Let

$$0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq 1$$
$$0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1,$$

be given constants such that $r_i < v_i$ for $i = 1, 2, \ldots, n$. If $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics in ascending order from a sample of n independent uniform random variables with ranges 0 to 1, then

$$Pr(r_i \leq X_{(i)} \leq s_i, 1 \leq i \leq n) = n! \det[(s_i - r_j)_+^{j-i+1}/(j-i+1)!]. \tag{7.7}$$

The difference between equations (7.1) and (7.7) is that in a $(\mathbf{r}, \mathbf{s})$-parking function, the sequence can only assume integer values. While a uniform random variable in $[0, 1]$ corresponds to *real-valued parking functions*, [2]. Hence equation (7.1) can be viewed as a discrete extension of the Steck formula (7.7).

The equation (7.4) can be extended to the sum-enumerator of $(\mathbf{r}, \mathbf{s})$-parking functions. Define

$$S_n(q; \mathbf{r}, \mathbf{s}) = \sum_{\alpha=(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n}$$

where the sum ranges over all $(\mathbf{r}, \mathbf{s})$-parking functions of length n. With a similar proof to that of (7.4), we can show that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \left((s_{i+1})_q - (r_n)_q\right)^{n-i} S_i(q; \mathbf{r}, \mathbf{s}) = 0,$$

where the factor $s_{i+1})_q - (r_n)_q$ is 0 if $r_n \geq s_{i+1}$. Hence, the sum-enumerator is a specialization of the polynomial $P_n(\mathbf{r}, \mathbf{s})$:

Theorem 7.2

$$S_n(q; \mathbf{r}, \mathbf{s}) = P_n(\mathbf{r}(\mathbf{q}), \mathbf{s}(\mathbf{q})),$$

where

$$\mathbf{r}(\mathbf{q}) = ((r_1)_q, (r_2)_q, \ldots, (r_n)_q),$$

and

$$\mathbf{s}(\mathbf{q}) = ((s_1)_q, (s_2)_q, \ldots, (s_n)_q).$$

7.2 Lattice paths with two-sided boundary

The number of lattice paths with two boundaries was obtained as a determinant formula by Steck [11, 12]. Such enumeration and various generalizations has been extensively studied, for example, in [4, Chapter2]. Hence we just list the main results on the subject, and explain the connection to biorthogonal polynomials.

Theorem 7.3 (Steck) *Let $a_0 \le a_1 \le \cdots \le a_m$ and $b_0 \le b_1 \le \cdots \le b_m$ be sequences of integers such that a_i, b_i. The number of sets of integers $(r_0, r_1, \ldots, r_m)$ such that $r_0 < r_1 < \cdots < r_m$ and $a_i < r_i < b_i$ for $0 \le i \le m$ is the $(m+1)$-th determinant $\det(d_{ij})$ where $d_{ij} = \binom{b_i - a_j + j - i - 1}{j - i + 1}$ if $i \le j$ and $b_i - a_j > 1$. Otherwise $d_{ij} = 0$.*

Denoted by $LP_n(\mathbf{a}, \mathbf{b})$ the number of lattice paths $(x_0, x_1, \ldots, x_{n-1})$ from $(0,0)$ to $(x-1, n)$ satisfying $a_i \le x_i < b_i < x$. Steck's formula gives

$$LP_n(\mathbf{a}, \mathbf{b}) = \det\left[\binom{(b_i - a_j)_+}{j - i + 1}\right]. \tag{7.8}$$

Formula (7.8) is a specialization of extended difference Goncarov polynomials. Given two sequences $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ and $\mathbf{b} = (b_0, b_1, b_2, \ldots)$, let $\tilde{g}_n^\dagger(x; \mathbf{a}, \mathbf{b}) = \tilde{g}_n^\dagger(x; a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1})$ $(n = 0, 1, 2 \ldots)$ be the sequence of polynomials biorthogonal to the operators

$$\psi_S(\Delta) = \Delta^s \sum_{r=0}^{\infty} (-1)^r \binom{(b_s - a_{s+r-1})_+}{r} \Delta^r. \tag{7.9}$$

Then

$$\tilde{g}_n^\dagger(0; \mathbf{a}, \mathbf{b})) = n! \det[\binom{(b_i - a_j)_+}{j - i + 1}] = n! LP_n(\mathbf{a}, \mathbf{b}). \tag{7.10}$$

These equations enable us to enumerate $LP_n(\mathbf{a}, \mathbf{b})$ from the theory of biorthogonal polynomials. Since generally, recurrence relations and generating functions are major techniques to solve a counting problem, we show how such results on $LP_n(\mathbf{a}, \mathbf{b})$ follow from the properties of $\tilde{g}_n^\dagger(0; \mathbf{a}, \mathbf{b}))$.

First, the linear recurrence (3.4) becomes

$$x^{(n)} = \sum_{i=0}^{n} \frac{n!}{i!} (-1)^{n-i} \binom{(b_i - a_{n-1})_+}{n - i} \tilde{g}_i^\dagger(x; \mathbf{a}, \mathbf{b})$$

It follows that

$$\delta_{0,n} = \sum_{i=0}^{n} (-1)^i \binom{(b_i - a_{n-1})_+}{n - i} \frac{1}{i!} \tilde{g}_i^\dagger(0; \mathbf{a}, \mathbf{b}) = \sum_{i=0}^{n} (-1)^i \binom{(b_i - a_{n-1})_+}{n - i} LP_i(\mathbf{a}, \mathbf{b}). \tag{7.11}$$

Equation (7.11) gives a linear recurrence to compute $LP_n(\mathbf{a}, \mathbf{b})$. This equation has been obtained in [4] as well, see equation (2.37). One can also prove it combinatorially by counting alternatively the set M_i of all pairs (α, i) where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an integer sequence satisfying (1) $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_i$, (2) $a_j \le \alpha_j < b_j$ for each $j = 0, 1, \ldots, i$, and (3) $\alpha_{i+1} < \alpha_{i+2} < \cdots < \alpha_n \in [a_n, b_{i+1})^{n-i}$. By a similar argument, if one define

$$\text{Area}_n(q; \mathbf{a}, \mathbf{b}) = \sum_{\mathbf{x}} q^{x_0+x_1+\cdots+x_{n-1}-a_0-\cdots-a_{n-1}},$$

Then

$$\delta_{0,n} = \sum_{i=0}^{n} (-1)^i \binom{(b_i - a_{n-1})_+}{n-i}_q \text{Area}_i(q; \mathbf{a}, \mathbf{b}).$$

From the Appell relation

$$\frac{1}{(1-t)^x} = \sum_{n=0}^{\infty} \tilde{g}_n^{\dagger}(x; \mathbf{a}, \mathbf{b}) \frac{\psi_n(t)}{n!},$$

we get the identity

$$\sum_{n=0}^{\infty} LP_n(\mathbf{a}, \mathbf{b}) \frac{\psi_n(t)}{n!} = 1,$$

where $\psi_n(t)$ is given in (7.9). In particular, when $a_i = ki + c$, $b_i = ki + d$ with $c < d$, i.e., lattice paths are restricted in a strip of width $d - c$, $\psi_n(t) = t^n f(t)$ where $f(t)$ is a polynomials of degree $\lceil \frac{d-c}{k} \rceil$. Hence the sequence $LP_n(\mathbf{a}, \mathbf{b})$ has a rational generating function

$$\sum_{n=0}^{\infty} LP_n(\mathbf{a}, \mathbf{b}) \frac{t^n}{n!} = \frac{1}{f(t)}.$$

It remains true even the initial boundaries a_i, b_i for $i = 0, 1, \ldots, T$ are arbitrary.

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