On the Enumeration of Non-crossing Pairings of Well-balanced Binary Strings

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Abstract

A non-crossing pairing on a binary string pairs ones and zeroes such that the arcs representing the pairings are non-crossing. A binary string is well-balanced if it is of the form $1^{a_1}0^{a_1}1^{a_2}0^{a_2}\dots 1^{a_r}0^{a_r}$. In this paper we establish connections between non-crossing pairings of well-balanced binary strings and various lattice paths in plane. We show that for well-balanced binary strings with $a_1 \leq a_2 \leq \cdots \leq a_r$, the number of non-crossing pairings is equal to the number of lattice paths on the plane with certain right boundary, and hence can be enumerated by differential Goncarov polynomials. For the regular binary strings $S = (1^k0^k)^n$, the number of non-crossing pairings is given by the $(k+1)$-Catalan numbers. We present a simple bijective proof for this case.

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Running title: Non-crossing pairings of binary strings

1 Introduction

Motivated by problems that arose from the theory of random matrices and free probability, Kemp, Mahlburg, Rattan, and Smyth[4] studied non-crossing pairings of binary strings. Given a binary string $S = 1^{a_1}0^{b_1}1^{a_2}0^{b_2}\cdots$, a non-crossing pairing on S matches ones and zeroes of S such that the arcs connecting the ones and zeroes are non-crossing if one puts the bits along a line and draws all the arcs above the bits. A binary string is *balanced*

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if the number of ones is the same as the number of zeroes. For a balanced binary string S, let $NC_2(S)$ be the set of non-crossing pairings of S. It is easy to see that $NC_2(S)$ is always non-empty. In [4], Kemp et.al. presented recurrence relations and functional equations satisfied by $|NC_2(S)|$, and proved general upper bounds in terms of the k-Catalan numbers. In this paper we examine a subset of this problem and further explore its properties.

Definition 1.1. *Let $S = 1^{a_1}0^{b_1}1^{a_2}0^{b_2}\cdots 1^{a_r}0^{a_r}$ be a binary string. We say that S is* well-balanced *if $a_i = b_i$ for all i. We say that S is* regular *if $a_i = b_j$ for all i, j.*

The main objective of this paper is to establish a connection between non-crossing pairings of well-balanced binary strings and lattice paths with a general right boundary. It is known in [4] that the non-crossing pairings for a regular binary string are counted by the k-Catalan numbers. In section 2 we give a bijective proof to complement the algebraic argument of [4]. We then list several lattice path representations for non-crossing pairings of regular binary strings. In section 3, we prove that for a well-balanced binary string $S = 1^{a_1}0^{a_1}\cdots 1^{a_{r+1}}0^{a_{r+1}}$ with $a_1 \leq a_2 \leq \cdots \leq a_{r+1}$, the number of non-crossing pairings can be computed as the number of lattice paths from $(0, 0)$ to (s_r, r) with the right boundary $(s_1, s_2, \ldots, s_r)$, where $s_i = a_1 + \cdots + a_i$ for $i = 1, 2, \ldots, r$. This result allows us to study $NC_2(S)$ by the powerful machinery in symbolic computation and umbral calculus. As corollaries, we give a determinant formula, a linear recurrence relation, and a (shifted) generating function for $|NC_2(S)|$. Finally in Section 4, we describe a bijective proof for the well-balanced strings.

2 Regular Binary Strings

The number of non-crossing pairings for a regular binary string $S = (1^k0^k)^n$ is equal to the nth $(k+1)$-Catalan number. This result was stated by Kemp et al. [4, Prop 1.6], where they outlined an algebraic proof analyzing the recurrence relations of both structures. In this section we present a simple bijective proof, mapping non-crossing pairings on a regular binary string to a k-Catalan structure.

The Catalan structure we will be using is the generalization of representation (n) of Stanley's online list [13]. The basic structure for $k = 2$ is n non-crossing chords pairing $2n$ (labeled) points on a circle. The k-Catalan version is non-intersecting regions in a circle, each containing k of the nk points on the circle. An equivalent definition is given in [1, item (10)], where it is described as a family of n non-intersecting chord paths of length $k-1$ joining kn points on the circumference of a circle. For each region in the first description, the chord path can be obtained by joining the points in this region in clockwise order. The chord paths are non-intersecting since they are contained in non-intersecting regions. Alternatively, one can create chord groups where the first point in the region (in clockwise order) is connected to each other point in the region with a chord, as in [12]. Again these chord groups are non-intersecting with each other.

All three representations are equivalent. See Fig. 1 for an illustration with $k = 3$ and $n = 2$, where the rightmost point on the circle is the initial point. In the following we shall use the first description in the bijection. The third one is convenient to extend to Catalan structures with a left weight, as demonstrated in [12].

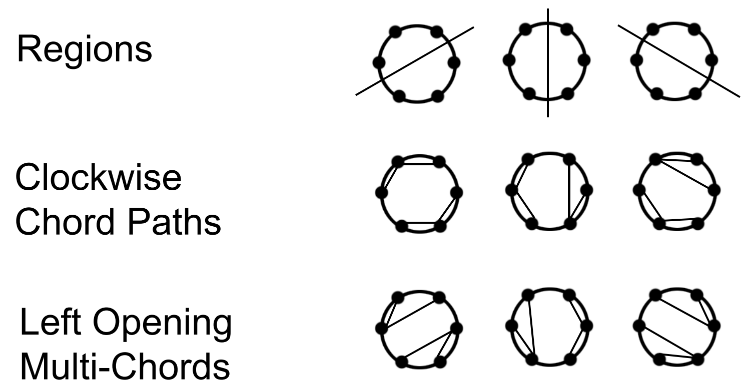

Figure 1: Three versions of representation (n) for the 2nd 3-Catalan number.

Theorem 2.1. *Let $S = (1^k0^k)^n$ be a regular binary string. Then $|NC_2(S)| = C_n^{k+1} = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}$, the nth $(k+1)$-Catalan number.*

Proof. We present a bijection between the set of non-crossing pairings for the binary string $S = (1^k0^k)^n$ and the set of ways to divide a circle inscribed with $(k+1)n$ points into n non-intersecting regions each with $k+1$ points. Given a non-crossing pairing of ones and zeroes of S, we first arrange the $2kn$ 01-bits clockwisely on a circle, starting from the position $(1, 0)$. That is, the first one of S corresponds to the rightmost point on the circle. The pairing is then represented by kn non-crossing chords of the circle. Next we compress each consecutive run of k ones to a single point, and hence obtain n points $Q_1, \ldots, Q_n$, where each Q_i is connected to k distinct zeroes on the circle with non-crossing chords. Now there are $(k+1)n$ points in total. Taking the convex hull of Q_i and its k neighbors for each i gives n non-intersecting regions.

The inverse map is slightly more complex. Given a circle with $n(k+1)$ points inscribed, divided into n distinct regions of $k+1$ points, start with the fixed starting point and mark the points in the positions $1, (k+1)+1, 2(k+1)+1, \ldots, (n-1)(k+1)+1$. We claim that each region contains exactly one marked point. To see this, it is enough to show that no two marked points can belong to the same region. Assume otherwise and the points P_i and P_j at positions $i(k+1)+1$ and $j(k+1)+1$ (where $i < j$) are in the same region A. Since the regions are non-intersecting, between any two consecutive points in the same region there are $(k+1)t$ points for some integer $t \geq 0$. Assume that there are y points lying strictly between P_i and P_j that are also in the region A. Then $0 \leq y \leq k-1$ and the total number of points between P_i and P_j is of the form $y + (k+1)t$. This leads to the equation $y + (k+1)t = (j-i)(k+1) - 1$, which implies that $y = -1 \pmod{k+1}$. This contradicts the fact that $0 \leq y \leq k-1$.

Now we expand each marked point into k points labeled by 1, and label each unmarked point by 0. Within each region there are $2k$ points, k with label 1 and k with 0. When reading clockwise, they are of the form $0^t1^k0^{k-t}$, which is equivalent to 1^k0^k by rotation and having exactly one non-crossing pairing. Laying all the points on a line and putting the non-crossing pairing for each region together gives us the non-crossing pairing of the appropriate regular binary string. □

As an example, in Fig. 2 we illustrate the correspondence between non-intersecting regions and non-crossing pairings of the regular binary string $(1^20^2)^2$.

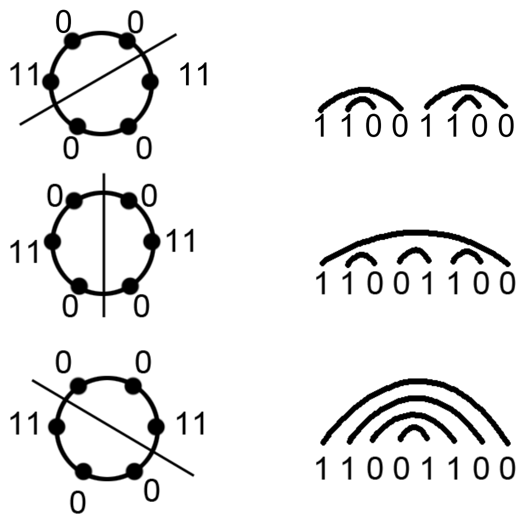

Figure 2: The corresponding regions and pairings of the binary string $(1^20^2)^2$.

The Catalan family contains many combinatorial structures, notably lattice paths. We list here some families of lattice paths that are also enumerated by the nth $(k+1)$-Catalan number, and hence equi-numerated with the non-crossing pairings of the regular binary strings. For all the cases of Corollary 2.2, bijective proofs are known: cases 1 and 2 are proved in [12], case 3 is given in [1] and case 4 in [6]. In the next section we will prove that lattice paths with a given right boundary plays an important role in the enumeration of non-crossing pairings of well-balanced binary strings. This result can be viewed as a generalization of case 4.

Corollary 2.2. *Let $S = (1^k0^k)^n$ be a regular binary string. Then the set of non-crossing pairings on S corresponds to the following sets of lattice paths.*

1. *Lattice paths from $(0,0)$ to (kn,kn) with steps $(0,k)$ and $(1,0)$, never rising above the line $y = x$. Equivalently, this is the set of sequences from $(0,0)$ to $(2kn,0)$ with steps (k,k) and $(1,-1)$, never falling below the x-axis.*

2. *Lattice paths from $(0,0)$ to $(2(kn+1)),0)$, starting with a $(1,1)$ step with the remaining steps in the path being n steps (k,k) and $kn+1$ steps $(1,-1)$, never falling below the x-axis, such that any maximal sequence of consecutive $(1,-1)$ steps ending on the x-axis has odd length.*

3. *Lattice paths from* $(0,0)$ *to* (kn,n) *with steps* $(1,0)$ *or* $(1,0)$*, never lying below the line* $x=ky$.

4. *Lattice paths from* $(0,0)$ *to* $(kn,n-1)$ *with steps* $(1,0)$ *or* $(1,0)$*, never touching the line* $x=ky+k+1$.

3 Well-balanced binary strings

For a well-balanced binary string $S = 1^{a_1}0^{a_1}1^{a_2}0^{a_2}\cdots 1^{a_r}0^{a_r}1^{a_{r+1}}0^{a_{r+1}}$, we say that S consists of $r+1$ segments, where the ith segment is $1^{a_i}0^{a_i}$. Again let $NC_2(S)$ be the set of non-crossing pairings of ones and zeroes of S. Following [4], we denote by $\phi^*(a_1,a_2,\ldots,a_r,a_{r+1})$ the cardinality of $NC_2(S)$.

In this section we prove that for any sequence of non-decreasing integers $a_1 \le a_2 \cdots \le a_r \le a_{r+1}$, $\phi^*(a_1,\ldots,a_{r+1})$ equals the number of lattice paths in the plane from $(0,0)$ to (x,r) with the right boundary $(s_1,s_2,\ldots,s_r)$, where $s_i = a_1+\cdots+a_i$. This result leads to many algebraic properties and explicit formulas for ϕ^*.

A lattice path P is a path in the plane with two kinds of steps: a unit north step $N=(0,1)$ or a unit east step $E=(1,0)$. If x is a positive integer, a lattice path from the origin $(0,0)$ to the point (x,n) can be coded by a length n non-decreasing sequence $(x_1,x_2,\ldots,x_n)$, where $0\le x_i\le x$ and x_i is the x-coordinate of the i-th north step. For example, the path $EENENNEE$ is coded by $(2,3,3)$. In other words, the i-th north step of the path has coordinates $(x_i,i-1)\to(x_i,i)$.

In general, let $\mathbf{s}$ be a non-decreasing sequence with positive integer terms s_1, $s_2,\ldots,$ s_n. A lattice path from $(0,0)$ to (x,n) is one with the (weak) right boundary $\mathbf{s}$ if $x_i\le s_i$ for $1\le i\le n$. If $x\ge s_n$, then the number of lattice paths from $(0,0)$ to (x,n) with the right boundary $\mathbf{s}$ does not depend on x. Let $Path_n(\mathbf{s})$ be the set of lattice paths from $(0,0)$ to (x,n) (where $x\ge s_n$) with the right boundary $\mathbf{s}$, and $LP_n(\mathbf{s})$ be the cardinality of $Path_n(\mathbf{s})$.

Given a sequence of positive integers $a_1\le a_2\le\cdots\le a_r\le a_{r+1}$, we first notice a basic fact.

Lemma 3.1. *$\phi^*(a_1,a_2,\ldots,a_r,a_{r+1})$ is independent of the value of a_{r+1}. That is,*

$$\phi^*(a_1,a_2,\ldots,a_r,a_{r+1}) = \phi^*(a_1,a_2,\ldots,a_r,a_r) \qquad \textit{for all } a_{r+1}\ge a_r. \tag{1}$$

Proof. To see this, for any $i>a_r$ let α be the ith one in the last segment $1^{a_{r+1}}0^{a_{r+1}}$. For any zero before α, it is easy to check that there are more ones than zeroes between this zero and α. Hence in any non-crossing pairing α must be paired with a zero in the last segment. There is only one way to do so, namely, we pair the last $a_{r+1}-a_r$ ones in the $(r+1)$st segment with the $a_{r+1}-a_r$ zeroes immediately after them. Hence (1) follows. $\square$

Clearly, $\phi^*(a)=1$. By convention, set $\phi^*(\emptyset)=1$. Our main result is the following equation of $\phi^*(a_1,a_2,\ldots,a_r,a_{r+1})$.

Theorem 3.2. *For a sequence of positive integers $a_1 \le a_2 \le \cdots \le a_r \le a_{r+1}$ with $r \ge 1$, let*

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ \cdots &= \cdots \\ s_r &= a_1 + a_2 + \cdots + a_r. \end{aligned}$$

Then $\phi^*(a_1, a_2, \ldots, a_r, a_{r+1})$ equals the number of lattice paths with the right boundary $(s_1, s_2, \ldots, s_r)$, that is

$$\phi^*(a_1, a_2, \ldots, a_r, a_{r+1}) = LP_r(s_1, s_2, \ldots, s_r). \tag{2}$$

We prove the theorem by a double induction on the length r and the first term a_1. First, when $r = 1$, the string S is of the form $1^{a_1}0^{a_1}1^{a_2}0^{a_2}$ where $a_1 \le a_2$. Given a non-crossing pairing, among the a_1 1s in the first segment, assume that i of them are paired with zeroes in the first segment, and $a_1 - i$ of them to zeroes in the second segment. There is only one way to arrange such a pairing. See Figure 3 for an illustration. As i ranges from 0 to a_1, we have $\phi^*(a_1, a_2) = a_1 + 1 = LP_1(a_1)$.

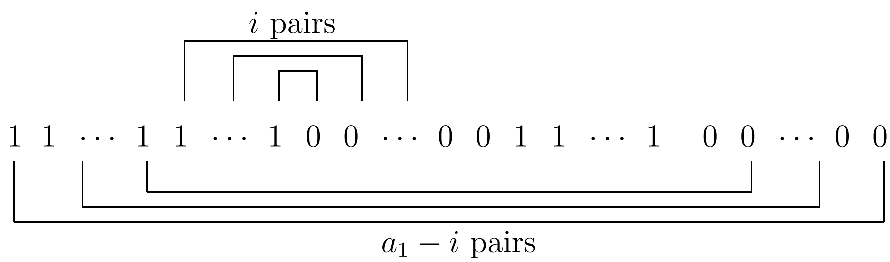

Figure 3: non-crossing pairings of $1^{a_1}0^{a_1}1^{a_2}0^{a_2}$.

Let us assume that Equation (2) holds for all sequences of length less than or equal to r. We shall show that it is true for sequences of length $r + 1$. To achieve this, we do induction on the first term a_1.

First we show Equation (2) for $a_1 = 1$. This is achieved by showing that $\phi^*(a_1, \ldots, a_{r+1})$ and $LP_r(s_1, \ldots, s_r)$ satisfy the same recurrence relation.

Lemma 3.3. *The number of non-crossing pairings on well-balanced strings satisfies the recurrence*

$$\phi^*(1, a_2, \ldots, a_r, a_{r+1}) = \sum_{i=1}^{r+1} \phi^*(a_2, \ldots, a_i)\phi^*(a_{i+1}, \ldots, a_{r+1}).$$

where we use the convention that $\phi^*(\emptyset) = 1$.

Proof. Each summand represents a choice of which segment to match the first zero in the string $S = 101^{a_2}0^{a_2} \cdots 1^{a_{r+1}}0^{a_{r+1}}$. If we match it to the one at the head of S, then we are left with the binary string $1^{a_2}0^{a_2} \cdots 1^{a_{r+1}}0^{a_{r+1}}$. There are $\phi^*(a_2, \cdots, a_{r+1})$ many non-crossing pairings in this case. Otherwise, we have to match the first zero to the very first one in the $(i+1)$st segment, for some $i = 1, 2, \ldots, r$. Call this pair the *first pair*. In that case, the string S is broken into two pieces, one is nested under the first pair with bits $1^{a_2}0^{a_2} \cdots 1^{a_i}0^{a_i}$, and the other is outside the first pair with bits $1^{a_{i+1}}0^{a_{i+1}} \cdots 1^{a_{r+1}}0^{a_{r+1}}$. They contribute $\phi^*(a_2, \ldots, a_i)\phi^*(a_{i+1}, \cdots, a_{r+1})$ non-crossing pairings. □

Next we prove that the number of lattice paths with the right boundary $(s_1, s_2, \ldots, s_r)$, where $s_1 = 1$, and $s_i = 1 + a_2 + \cdots + a_i$ for $i = 2, 3, \ldots, r$, satisfies a similar recurrence.

Lemma 3.4.

$$\begin{aligned} LP_r(1, s_2, \ldots, s_r) &= LP_{r-1}(s_2 - 1, s_3 - 1, \ldots, s_r - 1) + \\ &\sum_{i=1}^{r} LP_{i-2}(s_2 - 1, \ldots, s_{i-1} - 1) \cdot LP_{r-i}(a_{i+1}, a_{i+1} + a_{i+2}, \ldots, a_{i+1} + \cdots + a_r), \end{aligned}$$

where we use the convention that $LP_{-1} = LP_0 = 1$.

Proof. Let x be any number larger than s_r. For lattice paths from $(0, 0)$ to (x, r) with the right boundary $(s_1, s_2, \ldots, s_r)$, we partition them according to the minimal i such that the point $(s_i, i-1)$ is on the lattice path.

Case 1. The path does not contain the lattice point $(s_i, i-1)$ for any $i = 1, 2, \ldots, r$. Such a lattice path is strictly on the left of the boundary $(s_1, s_2, \ldots, s_r)$. They are counted by $LP_r(0, s_2 - 1, s_3 - 1, \ldots, s_r - 1) = LP_{r-1}(s_2 - 1, s_3 - 1, \ldots, s_r - 1)$.

Case 2. The lattice path does not pass the points $(s_1, 0), (s_2, 1), \ldots, (s_{i-1}, i-2)$, but contains $(s_i, i-1)$. Then the part from $(0, 0)$ to $(s_i, i-1)$ is strictly on the left of $(s_1 = 1, s_2, \ldots, s_{i-1})$ and is counted by $LP_{i-1}(0, s_2 - 1, s_3 - 1, \ldots, s_{i-1} - 1)$, which equals $LP_{i-2}(s_2 - 1, \ldots, s_{i-1} - 1)$ when $i \geq 3$, and is 1 for $i = 1, 2$. The next step on the path must be an N-step from $(s_i, i-1)$ to (s_i, i). The part from (s_i, i) to (x, r) is bounded by $(s_{i+1}, s_{i+2}, \ldots, s_r)$ and is counted by $LP_{r-i}(s_{i+1} - s_i, s_{i+2} - s_i, \ldots, s_r - s_i)$. Summing over Case 2 with $i = 1, 2, \ldots, r$ and combining with Case 1 gives the desired recurrence formula. □

By the inductive hypothesis, we have that

$$LP_{r-1}(s_2 - 1, \ldots, s_r - 1) = \phi^*(s_2 - 1, \ldots, s_r - s_{r-1}, *) = \phi^*(a_2, \ldots, a_r, a_{r+1}).$$

Similarly,

$$\begin{aligned} LP_{i-2}(s_2 - 1, \ldots, s_{i-1} - 1) &= \begin{cases} 1 & \text{if } i = 1 \\ \phi^*(a_2, \ldots, a_i) & \text{for } i \geq 2 \end{cases} \\ LP_{r-i}(s_{i+1} - s_i, s_{i+2} - s_i, \ldots, s_r - s_i) &= \phi^*(a_{i+1}, \ldots, a_r, a_{r+1}). \end{aligned}$$

Hence Lemmas 3.3 and 3.4 imply Equation (2) for $a_1 = 1$.

Now assume Equation (2) is true for all non-decreasing integer sequences of length less than or equal to r, and true for all non-decreasing sequences of length $r + 1$ with the first term less than a_1. We shall show it is true for the sequence $a_1 \le a_2 \le \cdots \le a_r \le a_{r+1}$. This is again achieved by showing that $\phi^*(a_1, \ldots, a_{r+1})$ and $LP_r(s_1, \ldots, s_r)$ satisfy the same recurrence relation. We note that Lemma 3.5 has been given in [4, Thm 3.3]. Since the argument is short and parallel to that of Lemma 3.3, we include it here for completeness.

Lemma 3.5. *Let $a_1 > 1$. We have*

$$\phi^*(a_1, \ldots, a_r, a_{r+1}) = \sum_{i=1}^{r+1} \phi^*(a_1 - 1, a_2, \ldots, a_i)\phi^*(a_{i+1}, \ldots, a_{r+1}). \tag{3}$$

Proof. Let α be the first zero in the binary string $S = 1^{a_1}0^{a_1} \cdots 1^{a_{r+1}}0^{a_{r+1}}$. We partition the non-crossing pairings of the zeroes and ones of S according to the one that pairs with α.

Case 1. α is paired with the last 1 in the first segment. Such non-crossing pairings are counted by $\phi^*(a_1 - 1, a_2, \ldots, a_{r+1})$. This is the last term in the summation of (3).

Case 2. α is paired with the bit β, which is a 1 in the $(i+1)$th segment, where $1 \le i \le r$. Then β must be at the a_1 position in the $(i+1)$th segment. Hence the string S is broken into two pieces, one is nested under the pair (α, β) with bits $0^{a_1-1}1^{a_2}0^{a_2} \cdots 1^{a_i}0^{a_i}1^{a_1-1}$ which is rotated to the string $1^{a_1-1}0^{a_1-1}1^{a_2}0^{a_2} \cdots 1^{a_i}0^{a_i}$, and the other is outside the pair (α, β) with bits $1^{a_{i+1}}0^{a_{i+1}} \cdots 1^{a_{r+1}}0^{a_{r+1}}$. This gives the product

$$\phi^*(a_1 - 1, a_2, \ldots, a_i)\phi^*(a_{i+1}, \ldots, a_{r+1})$$

for $1 \le i \le r$.

Combining the above two cases, we obtain the recurrence (3). □

Let $s_i = a_1 + \cdots + a_i$ for $i = 1, 2, \ldots, r$. Next we consider lattice paths with the right boundary $(s_1, s_2, \ldots, s_r)$.

Lemma 3.6. *For $s_1 > 1$, we have*

$$LP_r(s_1, s_2, \ldots, s_r) = \sum_{i=1}^{r+1} LP_{i-1}(s_1 - 1, \ldots, s_{i-1} - 1) \cdot LP_{r-i}(a_{i+1}, a_{i+1} + a_{i+2}, \ldots, a_{i+1} + \cdots + a_r). \tag{4}$$

Proof. Again we partition the lattice paths from $(0, 0)$ to (x, r) (where $x \ge s_r$)) with the right boundary $(s_1, s_2, \ldots, s_r)$ by the minimal i such that the path contains the point $(s_i, i - 1)$.

Case 1. The path does not contain any of the points $\{(s_i, i - 1) : i = 1, 2, \ldots, r\}$. There are $LP_r((s_1 - 1, s_2 - 1, \ldots, s_r - 1)$ many such paths. This gives the last term in the summation of (4).

Case 2. The path does not contain $(s_1, 0), \ldots, (s_{i-1}, i-2)$ but contains $(s_i, i-1)$. Then the part from $(0, 0)$ to $(s_i, i - 1)$ is strictly on the left of $(s_1, s_2, \ldots, s_{i-1})$ and is counted by $LP_{i-1}(s_1 - 1, s_2 - 1, \ldots, s_{i-1} - 1)$. The next step on the path must be an N-step from

$(s_i, i-1)$ to (s_i, i). The part from (s_i, i) to (x, r) is bounded by $(s_{i+1}, \ldots, s_r)$, and is counted by $LP_{r-i}(s_{i+1} - s_i, s_{i+2} - s_i, \ldots, s_r - s_i)$. Noting that $s_j - s_i = a_{i+1} + \cdots + a_j$, we obtain the recurrence (4). $\square$

Theorem 3.2 follows from the recurrences (3) and (4) and the inductive hypothesis. $\square$

One advantage of equating a new combinatorial structure to lattice paths is that lattice paths are a classical subject in combinatorial theory, whose enumeration can been systematically approached by symbolic computation and umbral calculus, see, for example, the classical book by Mohanty [6] and the extensive work done by H. Niederhausen and his collaborators [7, 8, 9, 10, 11, 2, 3]. In particular, it is known that lattice paths within general boundaries can by computed by a determinant formula. See Theorem 1 of [6, p.32]. Applying Theorem 3.2, we obtain the following determinant formula for the value of $\phi^*(a_1, \ldots, a_r, a_{r+1})$.

Corollary 3.7. *Let $1 \le a_1 \le a_2 \le \cdots \le a_r \le a_{r+1}$ be a non-decreasing integer sequence of length $r+1$. Then*

$$\phi^*(a_1, a_2, \ldots, a_{r+1}) = \det M,$$

where M is a $r \times r$ matrix whose entries are given by $M_{i,j} = \binom{s_i+1}{j-i+1}$.

We check some easy cases.

1. For $r = 1$, $s_1 = a_1$, and any $a_2 \ge a_1$, we have $\phi^*(a_1 a_2) = \det_{1\times 1}(a_1 + 1) = a_1 + 1$.

2. For $r = 2$, $s_1 = a_1$ and $s_2 = a_1 + a_2$, for any $a_1 \le a_2 \le a_3$ we have

$$\begin{aligned} \phi^*(a_1, a_2, a_3) &= \det \begin{pmatrix} s_1 + 1 & \binom{s_1+1}{2} \\ 1 & s_2 + 1 \end{pmatrix} \\ &= (s_1 + 1)(s_2 + 1 - \frac{s_1}{2}) \\ &= (a_1 + 1)(a_2 + 1 + \frac{a_1}{2}). \end{aligned}$$

3. For the regular binary strings $S = (1^k 0^k)^n$,

$$\phi^*(k, k, \ldots, k) = LP_{n-1}(k, 2k, \ldots, (n-1)k)$$

is the number of lattice paths from $(0,0)$ to $(x, n-1)$ (where $x > (n-1)k$) that never touches the line $x = ky + k + 1$. Using the formula (1.11) of [6, p.9], we have

$$\begin{aligned} |NC_2((1^k 0^k)^n)| &= \frac{k+1}{k+1+(k+1)(n-1)} \binom{k+1+(k+1)(n-1)}{n-1} \\ &= \frac{1}{(k+1)n+1} \binom{(k+1)n+1}{n}, \end{aligned}$$

which is the n-th $(k+1)$-Catalan number.

Another approach in lattice path counting is to use the analytical theory of biorthogonal polynomials. In [5] the second author, together with Kung and Sun, proved that $LP_r(s_1, \dots, s_r)$ can be represented by differential Goncarov polynomials, which is a sequence of polynomials that are biorthogonal to a given sequence of linear operators, each of which is a formal power series of the (backward) difference operator. Explicitly, let $\tilde{g}_r(x; b_1, \dots, b_r)$ be the difference Goncarov polynomials with the variable x and parameters $b_1, \dots, b_r$. Then Theorem 4.1 of [5] states that

$$LP_r(s_1, \dots, s_r) = \frac{1}{r!}\tilde{g}_r(x; x - s_1 - 1, x - s_2 - 1, \dots, x - s_r - 1), \forall x \in \mathbb{R}.$$

In particular, letting $x = 1$, we derive the following equation.

Corollary 3.8. *Let $a_1 \le a_2 \le \cdots \le a_{r+1}$ and $s_i = a_1 + \cdots + a_i$. Then*

$$\phi^*(a_1, \dots, a_r, a_{r+1}) = \frac{1}{r!}\tilde{g}_r(1; -s_1, -s_2, \dots, -s_r). \tag{5}$$

Goncarov polynomials bear many nice analytical and combinatorial properties, including the expansion formula, a linear recurrence, the Appell relations, difference relations, summation formula, shift invariant formula, and binomial expansion. See [5] for an expository article on the theory of Goncarov polynomials and its application in combinatorics. Our result in the present paper allows one to apply these powerful mechanisms to the enumeration of non-crossing pairings of well-balanced binary strings. As examples, we list the linear recurrence and the Appell relation of differential Goncarov polynomials, which give a linear recurrence and a shifted generating function for ϕ^*. In fact, many of the combinatorial properties of ϕ^* discussed in [4] can be deduced from the known results of lattice paths and Goncarov polynomials.

- Linear Recurrence for Differential Goncarov Polynomials.

$$x^{(n)} = \sum_{i=0}^{n} \binom{n}{i} b_i^{(n-i)} \tilde{g}_i(x; b_0, b_1, \dots, b_{i-1}) \tag{6}$$

  where $x^{(n)}$ is the rising factorial $x^{(n)} = x(x+1)\cdots(x+n-1)$.

- Appell relation for Differential Goncarov Polynomials.

$$(1-t)^{-x} = \sum_{n=0}^{\infty} \tilde{g}_n(x; b_0, \dots, b_{n-1}) \frac{t^n}{n!(1-t)^{b_n}}. \tag{7}$$

Letting $x = 1$ we have the following results for $\phi^*(a_1, \dots, a_{r+1})$ with $a_1 \le a_2 \le \cdots \le a_{r+1}$.

Corollary 3.9. *Linear recurrence for ϕ^**

$$1 = \sum_{i=0}^{n} (-1)^{n-i} \frac{(s_i)_{(n-i)}}{(n-i)!} \phi^*(a_1, a_2, \dots, a_{i+1}),$$

where $(x)_n$ is the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$.

Corollary 3.10. *Appell relation for* ϕ^*.

$$\sum_{i=0}^{\infty} t^i(1-t)^{s_{i+1}}\phi^*(a_1,\ldots,a_{i+1}) = \frac{1}{1-t},$$

where $s_i = a_1 + \cdots + a_i$.

We remark that Theorem 3.2 is only proved for non-decreasing sequences $a_1 \le a_2 \le \cdots \le a_{r+1}$. It is easy to check that in general,

$$\phi^*(a_1,\ldots,a_i,a_{i+1},\ldots,a_r) \ne \phi^*(a_1,\ldots,a_{i+1},a_i,\ldots,a_r).$$

Finding explicit formulas for $\phi^*(a_1,\ldots,a_r)$ for general integer sequences is still an open problem.

4 A bijective proof for Theorem 3.2

Inspired by the recurrence proof in Section 3, we construct a bijection between the set of non-crossing pairings of a well-balanced binary string to the set of lattice paths with appropriate right boundary. The bijection is established via an intermediate object– the ordered left-weighted ballot sequences, or the ordered left-weighted parenthesization, which are special cases of the left-weighted Catalan extensions studied and cataloged by the first author in [12].

Definition 4.1. *Given a sequence of positive integers* $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ *such that* $\sum_{i=1}^k b_i = n$*, an* ordered ballot sequence of left-weight $\mathbf{b}$ *is an arrangement of* $b_1, b_2, \ldots, b_k$ *and* n *copies of* -1 *such that (i) the terms* $b_1,\ldots,b_k$ *appear in order, and (ii) the partial sums are always non-negative.*

For example, the ordered ballot sequences of left-weight $(2,1)$ are listed below.

$$21--- \qquad 2-1-- \qquad 2--1-$$

An equivalent description is the ordered left-weighted parenthesization, where each integer b_i represents a left parenthesis of weight b_i, (or an inseparable block of b_i many left-parentheses), and each -1 represents one right-parenthesis. For each b_i, we draw a line from b_i to the last right-parenthesis that matches with it, and refer to it as the *region* of the left-parenthesis b_i. See the following figure for an illustration, where we also indicate which right parentheses are matched with b_i.

Let $a_1 \le a_2 \le \cdots \le a_r \le a_{r+1}$ be a sequence of positive integers, and let $s_i = a_1 + \cdots + a_i$ for $i = 1, 2, \ldots, r$. We will use $LwP(a_1,\ldots,a_r)$ to denote the set of ordered left-weighted parentheses with the weight $(a_1,\ldots,a_r)$. Let $S = 1^{a_1}0^{a_1}\cdots 1^{a_{r+1}}0^{a_{r+1}}$. We shall construct two bijections, bijection f from $Path_r(s_1,\ldots,s_r)$ to $LwP(a_1,\ldots,a_r,a_{r+1})$ and bijection g from $LwP(a_1,\ldots,a_{r+1})$ to $NC_2(S)$. We will then combine f and g to create a bijection from $Path_r(s_1,\ldots,s_r)$ to $NC_2(S)$

Bijection f **from** $Path_r(s_1,\ldots,s_r)$ **to** $LwP(a_1,\ldots,a_r,a_{r+1})$**.**

For any path P from $(0,0)$ to (s_r, r) with the right boundary $s_1,\ldots,s_r$, write the path P as a sequence of E and Ns. There are r N-steps and s_r many E steps.

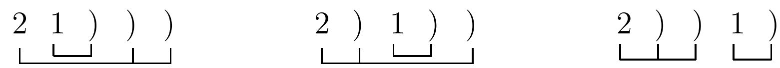

Figure 4: Ordered left-weighted parentheses and the corresponding regions

1. Add an N at the beginning of P, and a_{r+1} many E at the end of P.
2. Replace the ith N with a left-parenthesis of weight a_i.
3. Replace each E with a right parenthesis.

For example, let the positive integers a_i be $2, 2, 3, 4$, then the right boundary is $\mathbf{s} = (2, 4, 7)$. For a path P coded by $(1, 1, 2)$, the steps in P are $ENNENEEEEE$, then $f(P) = 2)23)4)))))))))$.

It is easy to see that $f(P) \in LwP(a_1, \ldots, a_r, a_{r+1})$ whenever $P \in Path_r(s_1, \ldots, s_r)$. The steps are easy to reverse, so f is a bijection.

Bijection g from $LwP(a_1, \ldots, a_{r+1})$ to $NC_2(S)$, where $S = 1^{a_1}0^{a_1} \cdots 1^{a_{r+1}}0^{a_{r+1}}$ and $a_1 \leq a_2 \leq \cdots \leq a_{r+1}$.

The map g is defined inductively. First set $g(\emptyset) = \emptyset$. For $r = 0$, the unique element of $LwP(a)$ is mapped to the unique non-crossing pairing of 1^a0^a.

Assume that g is a well-defined bijection from the set $LwP(a_1, \ldots, a_l)$ to the set $NC_2(1^{a_1}0^{a_1} \cdots 1^{a_l}0^{a_l})$ for all $l \leq r$ and $a_1 \leq a_2 \leq \cdots \leq a_l$. We shall define g on $LwP(a_1, \ldots, a_{r+1})$.

Given an ordered left-weighted parenthesization $\pi \in LwP(a_1, \ldots, a_{r+1})$, let $\alpha_1, \ldots, \alpha_{a_1}$ be the right parentheses that match the first left parenthesis (of weight a_1). These are the right parentheses that belong to the region of the first left parenthesis but not to any other regions.

We count the left parentheses from left to right. Let c_i be the number of left parentheses before α_i (with the convention that $c_0 = 1$, α_0 is the first left parenthesis, and $\alpha_{a_1+1} = \infty$). Then the segment π_i, between α_i and α_{i+1} (for $i = 0, 1, \ldots, a_1$), will consist of the left parentheses numbered $c_i + 1$ to c_{i+1} and their matching right parentheses. Thus, π_i is an ordered left-weighted parenthesization with the non-decreasing weight $(a_{c_i+1}, \ldots, a_{c_{i+1}})$.

By inductive hypothesis, $g(\pi_i)$ is a non-crossing pairing of the binary string

$$S_i = 1^{a_{c_i+1}}0^{a_{c_i+1}} \cdots 1^{a_{c_{i+1}}}0^{a_{c_{i+1}}}.$$

Let $\bar{S}_i$ be the string obtained from S_i by moving the last $a_1 - i$ zeroes to the beginning. Then $g(\pi_i)$ is also a non-crossing pairing of $\bar{S}_i$, as the non-crossing pairings are invariant under the rotation of the underlying binary string.

To construct $g(\pi)$, start with the unique pairing of $1^{a_1}0^{a_1}$. Denote by β_0 the substring 1^{a_1}, and β_i the ith zero. Let $\beta_{a_1+1} = \infty$. For $i = 0, 1, \ldots, a_1$, insert the string $\bar{S}_i$ between β_i and β_{i+1}, together with the pairing $g(\pi_i)$. This gives a non-crossing pairing of $1^{a_1}0^{a_1} \cdots 1^{a_{r+1}}0^{a_{r+1}}$.

The above steps are reversible, hence it defines a bijection.

As an example, the following figures show the non-crossing pairing corresponding to the ordered left-weighted parenthesization 2)23)4)))))))))).

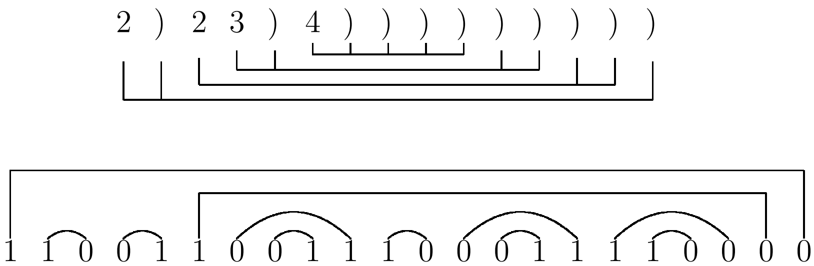

Figure 5: An ordered left-weighted parenthesization and the corresponding non-crossing pairing.

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