Maximal increasing sequences in fillings of almost-moon polyominoes

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Abstract

It was proved by Rubey that the number of fillings with zeros and ones of a given moon polyomino that do not contain a northeast chain of a fixed size depends only on the set of column lengths of the polyomino. Rubey's proof uses an adaption of jeu de taquin and promotion for arbitrary fillings of moon polyominoes and deduces the result for 01-fillings via a variation of the pigeonhole principle. In this paper we present the first completely bijective proof of this result by considering fillings of almost-moon polyominoes, which are moon polyominoes after removing one of the rows. More precisely, we construct a simple bijection which preserves the size of the largest northeast chain of the fillings when two adjacent rows of the polyomino are exchanged. This bijection also preserves the column sum of the fillings. In addition, we also present a simple bijection that preserves the size of the largest northeast chains, the row sum and the column sum if every row of the filling has at most one 1. Thereby, we not only provide a bijective proof of Rubey's result but also two refinements of it.

Keywords: maximal chains, moon polyominoes

MSC Classification: 05A19

1 Introduction

The systematic study of matchings and set partitions with certain restrictions on their crossings and nestings started in [3], where Chen, Deng, Du, Stanley, and Yan used Robinson-Schensted-like insertion/deletion processes to show the symmetry between the sizes of the largest crossings and the largest nestings. Lying in the heart of [3] is Greene's Theorem on the relation between increasing and decreasing subsequences in permutations and the shape of the tableaux which are obtained by the Robinson-Schensted correspondence. These results have been put in a larger context of enumeration of fillings of polyominoes where one imposes restrictions on the increasing and decreasing chains of the fillings.

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Moon polyominoes are polyominoes that are convex and intersection-free. In a moon polyomino $\mathcal{M}$ the lengths of rows (and columns) are arranged in a unimodal order. Let $\mathcal{F}_{\mathbb{N}}(\mathcal{M}, n)$ be the set of assignments of nonnegative integers to the cells of $\mathcal{M}$ such that the sum of the integers equals n and let $\mathcal{F}_{01}(\mathcal{M}, n)$ be the subset of $\mathcal{F}_{\mathbb{N}}(\mathcal{M}, n)$ where the cells may only be filled with zeros and ones. Let $\mathrm{ne}(M)$ be the length of the longest strict northeast chain in a filling M. The motivation for this paper is to better understand the striking result of Rubey [16] that the cardinalities of the sets $\{M : M \in \mathcal{F}_{\mathbb{N}}(\mathcal{M}, n) \text{ and } \mathrm{ne}(M) = k\}$ and $\{M : M \in \mathcal{F}_{01}(\mathcal{M}, n) \text{ and } \mathrm{ne}(M) = k\}$ only depend on the multiset of column lengths of the moon polyomino $\mathcal{M}$. This is a very interesting property for fillings of moon polyominoes: many combinatorial statistics have distributions which are invariant under permutations of rows (or columns). For example, a similar statement is also known for the major index introduced by Chen, Poznanović, Yan and Yang [4], for the numbers of northeast and southeast chains of length 2 studied by Kasraoui [13], and for various analogs and generalizations of 2-chains [5, 19] .

The aforementioned result of Rubey is a culmination of the work of many researchers on various subclasses of moon polyominoes. The case of Ferrers shapes was proved bijectively by Krattenthaler [14] using Fomin's growth diagrams [8, 9, 10]. Jonsson, motivated by the problem of counting generalized triangulations avoiding crossings of a given size, first proved the result for maximal 01-fillings of stack polyominoes using an involved induction [11], and later with Welker [12] for all 01-fillings with a fixed number of 1s using the machinery of simplicial complexes and commutative algebra. A stack polyomino is a convex polyomino in which the rows are arranged in a descending order from top to bottom. A bijective proof for the general case of $\mathbb{N}$-fillings of moon polyominoes was obtained by Rubey [16] using an adaptation of *jeu de taquin* and promotion. Rubey additionally deduced the statement for $\mathcal{F}_{01}(\mathcal{M}, n)$ using essentially the pigeonhole principle. Later, a bijective proof for $\mathcal{F}_{01}(\mathcal{M}, n)$ with the additional assumption that n is maximal was provided by Serrano and Stump [18] for the special case of Ferrers shapes and extended to stack polyominoes by Rubey [17].

In this paper we give a bijective proof of the Rubey's result for $\mathcal{F}_{01}(\mathcal{M}, n)$. More precisely, we construct two simple bijections for 01-fillings demonstrating two different refinements of this result. For a moon polyomino $\mathcal{M}$, let $\sigma\mathcal{M}$ be another moon polyomino obtained by permuting the rows of $\mathcal{M}$. The main idea in constructing the desired bijection is to extend the class of moon polyominoes to a more general family that would allow us to transform the moon polyomino $\mathcal{M}$ to any $\sigma\mathcal{M}$ by a sequence of steps that interchange two adjacent rows at each time. For this purpose we introduce the notion of almost-moon polyominoes, which become moon polyominoes after removing one of its rows (see Section 2 for the exact definition). Let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes that are related by an interchange of two adjacent rows. We present two bijections. The first is a map $\phi_{\mathcal{M},\mathcal{N}}$ from the 01-fillings of $\mathcal{M}$ to those of $\mathcal{N}$ which peserves the number of non-zero entries, the size of the longest northeast chain and the column sums. The second map $\psi_{\mathcal{M},\mathcal{N}}$ is restricted to fillings in which every row has at most one 1 and preserves the size of the longest northeast chains, the row sum, and the column sum. Both our maps are simple to describe but the proofs that they have the desired properties are technical.

Note that Rubey's result implies that the number of 01-fillings of a moon polyomino without northeast chains of size k equals those without southeast chains of size k. We should mention that there are combinatorial transformations and bijections between certain families of fillings that avoid northeast chains of size k and southeast chains of size k, for example, by Backelin, West and Xin [1] for 01-fillings of Ferrers shapes where every row and every column has exactly one 1, and by de Mier [7] for $\mathbb{N}$-fillings of Ferrers shapes with fixed row sum and column sum. However, none of them preserves the size of the longest northeast chains which is an important property of the bijections that we provide.

The paper is organized as follows. Section 2 contains the necessary notation and the statements of the main results. In Section 3 we construct the bijection $\phi_{\mathcal{M},\mathcal{N}}$ for fillings of almost-moon polyominoes which preserves the size of maximal northeast chains and the columns sum. In Section 4 we restrict ourselves to fillings that have at most one 1 in each row and describe the bijection $\psi_{\mathcal{M},\mathcal{N}}$. We conclude the paper with some comments and counterexamples to a few seemingly natural generalizations in Section 5.

2 Notation and statements of the main results

A *polyomino* is a finite subset of $\mathbb{Z}^2$, where we represent every element (i, j) of $\mathbb{Z}^2$ by a square cell. The polyomino is *row-convex* (*column-convex*) if its every row (column) is connected. If the polyomino is both row- and column-convex, we say that it is *convex*. It is *intersection-free* if every two columns are *comparable*, i.e., the row-coordinates of one column form a subset of those of the other column. Equivalently, it is intersection-free if every two rows are comparable. A *moon polyomino* is a convex intersection-free polyomino (e.g. Figure 1a). The length of a row (or a column) is the number of cells in it. Note that in a moon polyomino the lengths of rows from top to bottom form a unimodal sequence. We will say that a row $\mathcal{R}$ is an *exceptional row* of a polyomino $\mathcal{M}$ if there are rows above $\mathcal{R}$ and below $\mathcal{R}$ with larger lengths in $\mathcal{M}$. An *almost-moon polyomino* is a polyomino with comparable convex rows and at most one exceptional row (e.g. Figure 1b). Therefore, every moon polyomino is also an almost-moon polyomino and an almost-moon polyomino is not necessarily column-convex.

(a) A moon polyomino (b) An almost-moon polyomino

Figure 1: A moon polyomino and an almost-moon polyomino with an exceptional row $\mathcal{R}$ that differ by an interchange of adjacent rows.

In this paper we will consider polyominoes whose cells are filled with zeros and ones. A *northeast chain*, or shortly *ne-chain*, of size k in such a filling is a set of k cells $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ filled with 1s such that the $k \times k$ submatrix

$$\mathcal{G} = \{(i_r, j_s) : 1 \le r \le k, 1 \le s \le k\}$$

is contained in the polyomino (with no restriction on the filling of the other cells). See Figure 2 for an illustration. We will call the ne-chains of size k shortly k-chains. Note that in a moon polyomino $\mathcal{M}$, k 1-cells in a north-east direction satisfy the submatrix condition if and only if the corners (i_1, j_k) and (i_k, j_1) are contained in $\mathcal{M}$, which is equivalent to the condition that the whole rectangle determined by these corners is contained in $\mathcal{M}$. In an almost-moon polyomino, the submatrix condition is satisfied if and only if the vertices (i_1, j_k) and (i_k, j_1) either determine a rectangle which is completely contained in $\mathcal{M}$ or an almost-rectangle with one exceptional row contained in $\mathcal{M}$. In the latter case, the exceptional row does not contain any elements from the ne-chain.

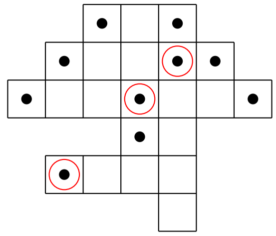

Figure 2: A 01-filling M of an almost-moon polyomino with $\mathrm{ne}(M) = 3$. The 1s are represented by dots and the 0-cells are drawn empty. The circled dots form the only 3-chain in M.

For a 01-filling M of an almost-moon polyomino, we denote by $\mathrm{ne}(M)$ the size of its largest ne-chains. Suppose that $\mathcal{M}$ has k rows and ℓ columns and let $\mathbf{r} \in \mathbb{N}^k$ and $\mathbf{c} \in \mathbb{N}^\ell$. Since we only consider 01-fillings in this paper, we will skip the subscript 01 and simply denote by $\mathcal{F}(\mathcal{M})$ the set of all 01-fillings of $\mathcal{M}$, by $\mathcal{F}(\mathcal{M}, n)$ those fillings with exactly n 1s, and by $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ the set of all 01-fillings with row sums given by $\mathbf{r}$ and column sums given by $\mathbf{c}$. Our first main result states that if $\mathcal{M}$ and $\mathcal{N}$ are two almost-moon polyominoes related by an interchange of adjacent rows (e.g. Figure 1), then the statistic ne is equidistributed over the sets $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$ and $\mathcal{F}(\mathcal{N}, *, \mathbf{c})$ of fillings of $\mathcal{M}$ and $\mathcal{N}$, respectively, with fixed column sums but arbitrary row sums:

$$\sum_{M \in \mathcal{F}(\mathcal{M}, *, \mathbf{c})} q^{\mathrm{ne}(M)} = \sum_{M \in \mathcal{F}(\mathcal{N}, *, \mathbf{c})} q^{\mathrm{ne}(M)}.$$

More precisely, we have the following theorem.

Theorem 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes such that $\mathcal{N}$ can be obtained from $\mathcal{M}$ by an interchange of two adjacent rows. In addition assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. Then there is a bijection*

$$\phi_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{N})$$

that preserves the column sums of the fillings and such that $\mathrm{ne}(\phi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M)$ for $M \in \mathcal{F}(\mathcal{M})$. Moreover, $\phi_{\mathcal{N},\mathcal{M}} \circ \phi_{\mathcal{M},\mathcal{N}} = 1_{\mathcal{F}(\mathcal{M})}$.

Let $\mathcal{M}$ be an almost-moon polyomino with k rows, $\sigma \in S_k$, and suppose the polyomino $\sigma\mathcal{M}$ obtained by permuting the rows of $\mathcal{M}$ according to σ is also an almost-moon polyomino. Note that $\sigma\mathcal{M}$ can also be obtained by a sequence of steps in which only two adjacent rows are interchanged. Moreover, the order of steps can be chosen so that the intermediate polyominoes are all almost-moon polyominoes with no exceptional rows other than the swapped ones. In other words, the set of all $\sigma\mathcal{M}$ which are almost-moon polyominoes is connected by transposition of adjacent rows. One way to see this is to note that one can reach the polyomino in which the row lengths are descending from top to bottom by starting from $\mathcal{M}$ and first moving its exceptional row down until there is no longer rows below it, then moving the shortest row of $\mathcal{M}$ to the bottom, the second shortest row to the second position from below, etc. Consequently, by composing the maps from Theorem 1 we get the following corollary.

Corollary 2. *Let $\mathcal{M}$ be an almost-moon polyomino with k rows and ℓ columns, $\sigma \in S_k$ be a permutation of the row indices such that $\sigma\mathcal{M}$ is also an almost-moon polyomino. Let $\boldsymbol{c} \in \mathbb{N}^\ell$. Then there is a bijection $\phi : \mathcal{F}(\mathcal{M}, *, \boldsymbol{c}) \longrightarrow \mathcal{F}(\sigma\mathcal{M}, *, \boldsymbol{c})$ such that $\mathrm{ne}(\phi(M)) = \mathrm{ne}(M)$ for $M \in \mathcal{F}(\mathcal{M}, *, \boldsymbol{c})$. Moreover, the size of $\{M : M \in \mathcal{F}(\mathcal{M}, n), \mathrm{ne}(M) = k\}$ depends only on the multiset of the column lengths of $\mathcal{M}$.*

Proof. To prove the second part of the statement, suppose $\mathcal{M}_1$ is the moon polyomino with descending row lengths that can be obtained by reordering the rows of $\mathcal{M}$ as in the discussion above. Suppose that the same procedure applied to the transpose $\mathcal{M}_1^t$ of $\mathcal{M}_1$ yields the polyomino $\mathcal{M}_2$ with descending row lengths. Then $\mathcal{M}_2$ is a Ferrers shape and it depends only on the multiset of the column lengths of $\mathcal{M}$. Since the transpose of ne-chains are also ne-chains, it follows from the first part that the size of $\{M : M \in \mathcal{F}(\mathcal{M}, n), \mathrm{ne}(M) = k\}$ also depends only on this multiset. ∎

The fact that $|\{M : M \in \mathcal{F}(\mathcal{M}, *, \mathbf{c}), \mathrm{ne}(M) = k\}| = |\{M : M \in \mathcal{F}(\sigma\mathcal{M}, *, \mathbf{c}), \mathrm{ne}(M) = k\}|$ if $\mathcal{M}$ and $\sigma\mathcal{M}$ are moon polyominoes as well as the second part of Corollary 2 was proved using a variation of the pigeonhole principle by Rubey [16]. Therefore, our results provide a bijective proof of these facts and extend them to a larger set of polyominoes in which these properties hold. In Section 5 we discuss why this extension is in a sense the best possible.

As discussed in [16], one cannot hope to simultaneously preserve both $\mathbf{r}$ and $\mathbf{c}$, i.e., the natural generalization $|\{M : M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c}), \mathrm{ne}(M) = k\}| = |\{M : M \in \mathcal{F}(\sigma\mathcal{M}, \sigma\mathbf{r}, \mathbf{c}), \mathrm{ne}(M) = k\}|$ does not hold. However, our second main result implies that ne can be preserved together with both the row and column sums if the fillings are restricted to have at most one 1 in each row.

Theorem 3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes such that $\mathcal{N}$ can be obtained from $\mathcal{M}$ by an interchange of two adjacent rows. In addition assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. If $\boldsymbol{r} \in \{0, 1\}^*$ and $\mathbf{c} \in \mathbb{N}^*$, then there is a map*

$$\psi_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}) \longrightarrow \mathcal{F}(\mathcal{N}, \boldsymbol{r}', \boldsymbol{c})$$

such that $\mathrm{ne}(\psi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M)$ for $M \in \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c})$, where $\mathbf{r}'$ is obtained from $\mathbf{r}$ by exchanging the entries corresponding to the two swapped rows.

This theorem was proved for moon polyominoes by Rubey [16]. By the same discussion after Theorem 1 and a combination of the maps $\psi_{\mathcal{M},\mathcal{N}}$ and $\phi_{\mathcal{M},\mathcal{N}}$, we get the following corollary.

Corollary 4. *Let $\mathcal{M}$ be an almost-moon polyomino with k rows and ℓ columns, $\sigma \in S_k$ be a permutation of the row indices such that $\sigma\mathcal{M}$ is also an almost-moon polyomino. Let $\boldsymbol{r} \in \{0, 1\}^k$ and $\mathbf{c} \in \mathbb{N}^l$. Then there is a bijection $\psi : \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}) \longrightarrow \mathcal{F}(\sigma\mathcal{M}, \sigma\boldsymbol{r}, \boldsymbol{c})$ such that $\mathrm{ne}(\psi(M)) = \mathrm{ne}(M)$ for $M \in \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c})$. Moreover, the number of fillings $M \in \mathcal{F}(\mathcal{M}, n)$ with $\mathrm{ne}(M) = k$ and at most one 1 per row depends only on the multiset of the column lengths of $\mathcal{M}$.*

3 Maximal increasing sequences in 01-fillings with fixed total sum

In this section we will describe the maps $\phi_{\mathcal{M},\mathcal{N}}$ and prove Theorem 1. To this end, let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes related by an interchange of two adjacent rows (e.g. Figure 1). Assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. If the swapped rows are of equal length then $\mathcal{M} = \mathcal{N}$ and we define $\phi_{\mathcal{M},\mathcal{N}}$ to be the identity map on $\mathcal{F}(\mathcal{M})$.

Otherwise, suppose the lengths of the swapped rows are not equal and let $\mathcal{R}_s$ and $\mathcal{R}_l$, respectively, be the shorter and longer rows. Note that $\mathcal{M}\backslash\mathcal{R}_s$ and $\mathcal{N}\backslash\mathcal{R}_s$ have no exceptional rows. Let $\alpha, \beta, \gamma, \delta$ be the fillings of the regions of $\mathcal{R}_s$ and $\mathcal{R}_l$ in M as depicted on the left side of Figure 3. Precisely, α is the filling of the shorter row $\mathcal{R}_s$, β is the filling of the part of the longer row $\mathcal{R}_l$ which has the same column support as row $\mathcal{R}_s$, and γ and δ are the fillings of the two ends of row $\mathcal{R}_l$ so that the whole filling of row $\mathcal{R}_l$ viewed as a binary string is a concatenation of γ, β, and δ. Note that one of γ and δ may be the empty string. First we define a map $f_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{N})$ as follows.

Definition 5. *Using the notation above, $f_{\mathcal{M},\mathcal{N}}(M)$ is the filling of $\mathcal{N}$ in which*

(1) the filling of the rows in $\mathcal{N}$ other than $\mathcal{R}_s$ and $\mathcal{R}_l$ is the same as in M,

(2) the filling of the shorter row $\mathcal{R}_s$ of $\mathcal{N}$ is β,

(3) the filling of the longer row $\mathcal{R}_l$ of $\mathcal{N}$ is the concatenation of γ, α, and δ.

See Figure 3 for an illustration.

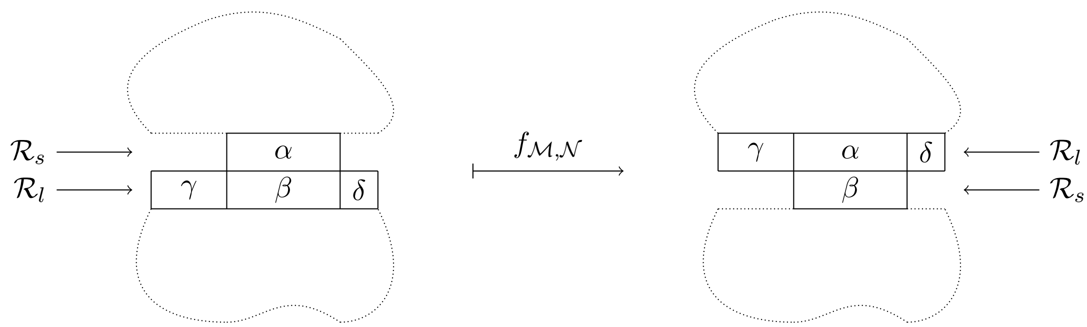

Figure 3: The fillings M and $f_{\mathcal{M},\mathcal{N}}(M)$ differ only in the rows $\mathcal{R}_s$ and $\mathcal{R}_l$.

It follows directly from the definition.

Lemma 6. $f_{\mathcal{N},\mathcal{M}} \circ f_{\mathcal{M},\mathcal{N}} = 1_{\mathcal{F}(\mathcal{M})}$.

If it is clear what the polyominoes $\mathcal{M}$ and $\mathcal{N}$ are, we will leave out the subscripts and write only $f(M)$.

Lemma 7. *If the almost-moon polyominoes $\mathcal{M}$ and $\mathcal{N}$ are related by an interchange of the rows $\mathcal{R}_s$ and $\mathcal{R}_l$ as above, then for every $M \in \mathcal{F}(\mathcal{M})$,*

$$|\text{ne}(M) - \text{ne}(f_{\mathcal{M},\mathcal{N}}(M))| \leq 1.$$

Proof. Let C be a k-chain in M.

(i) If C contains no 1-cells from $\mathcal{R}_s \cup \mathcal{R}_l$ then C is a k-chain in $f(M)$.

(ii) If C contains a 1-cell from α then C is a k-chain in $f(M)$.

(iii) If C contains a 1-cell b_0 from β then $C - \{b_0\}$ is a $(k-1)$-chain in $f(M)$.

(iv) If C contains a 1-cell from γ or δ then C is a k-chain in $f(M)$.

Therefore, $\mathrm{ne}(f(M)) \geq \mathrm{ne}(M) - 1$. By switching the roles of M and $f(M)$ with a similar analysis, we get $\mathrm{ne}(M) \geq \mathrm{ne}(f(M)) - 1$. ∎

Therefore, every filling $M \in \mathcal{F}(\mathcal{M})$ satisfies exactly one of the following 3 conditions:

$$\text{(I)}\quad \mathrm{ne}(f(M)) = \mathrm{ne}(M) \qquad \text{(II)}\quad \mathrm{ne}(f(M)) = \mathrm{ne}(M) + 1 \qquad \text{(III)}\quad \mathrm{ne}(f(M)) = \mathrm{ne}(M) - 1.$$

Let $\mathcal{F}^I(\mathcal{M})$, $\mathcal{F}^{II}(M)$, and $\mathcal{F}^{III}(\mathcal{M})$, be the fillings of $\mathcal{M}$ that satisfy the conditions (I), (II), and (III), respectively. Below we describe how $\phi_{\mathcal{M},\mathcal{N}}(M)$ is defined on each of these three sets.

Case I. For $M \in \mathcal{F}^I(\mathcal{M})$ we define $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(M)$. It is clear that this defines a bijection from $\mathcal{F}^I(\mathcal{M})$ onto $\mathcal{F}^I(\mathcal{N})$.

Case II. For $M \in \mathcal{F}^{II}(\mathcal{M})$ with $\mathrm{ne}(M) = k$, we have $\mathrm{ne}(N') = k + 1$ where $N' = f(M)$. Reasoning as in the proof of Lemma 7, one can see that all $(k+1)$-chains in N' contain exactly one cell from $\mathcal{R}_s \cup \mathcal{R}_l$ and that that cell must be in the part α of the longer row $\mathcal{R}_l$ of $\mathcal{N}$ (see the right filling in Figure 3).

Definition 8. *Suppose $M \in \mathcal{F}^{II}(\mathcal{M})$ and $\mathrm{ne}(f(M)) = \mathrm{ne}(M) + 1 = k + 1$. The 1-cells in row $\mathcal{R}_l$ of $f(M)$ which are part of a $(k+1)$-chain are called* problem cells.

Let α_0 be the set of problem cells in $N' = f(M)$ and let β_0 be the set of cells in row $\mathcal{R}_s$ of N' that share a horizontal edge with the problem cells. We define $\phi_{\mathcal{M},\mathcal{N}}(M)$ to be the filling N'' of $\mathcal{N}$ obtained by replacing the problem cells in α_0 by zeros and the cells in β_0 by ones. In other words, $\phi_{\mathcal{M},\mathcal{N}}(M) = N''$ is obtained by a vertical shift of the problem cells in N' from row $\mathcal{R}_l$ to row $\mathcal{R}_s$.

Case III. For $M \in \mathcal{F}^{III}(\mathcal{M})$, we have $f_{\mathcal{M},\mathcal{N}}(M) \in \mathcal{F}^{II}(\mathcal{N})$. In this case we set $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(\phi_{\mathcal{N},\mathcal{M}}(f_{\mathcal{M},\mathcal{N}}(M)))$.

In the following sequence of lemmas we show that $\phi_{\mathcal{M},\mathcal{N}}$ is a well-defined bijection from $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$ to $\mathcal{F}(\mathcal{N}, *, \mathbf{c})$ preserving the statistic ne. Lemmas 9–12 show that $\phi_{\mathcal{M},\mathcal{N}}$ defined in Case II preserves column sum and $\mathrm{ne}(N'') = k$. Lemma 13 shows that $\phi_{\mathcal{M},\mathcal{N}}$, when restricted to $\mathcal{F}^{II}(\mathcal{M})$, is a bijection onto $\mathcal{F}^{II}(\mathcal{N})$. Lemma 14 shows $\phi_{\mathcal{M},\mathcal{N}}$ restricted to $\mathcal{F}^{III}(\mathcal{M})$ is a bijection onto $\mathcal{F}^{III}(\mathcal{N})$.

We first assume that M is a filling in $\mathcal{F}^{II}(\mathcal{M})$ with $\mathrm{ne}(M) = k$.

Lemma 9. *If a_0 in row $\mathcal{R}_l$ of $f(M)$ is a problem cell then the cell b_0 in row R_s of $f(M)$ in the same column as a_0 is a 0-cell.*

Proof. Suppose a_0 is part of the $(k+1)$-chain C in $f(M)$. If b_0 is a 1-cell then $(C - \{a_0\}) \cup \{b_0\}$ is a $(k+1)$-chain in M. ∎

Lemma 9 shows that in Case II, the cells in β_0 are 0-cells for N'. Hence N'' is a 01-filling of $\mathcal{N}$ and the column sums of M and N'' are the same. We will prove next that $\mathrm{ne}(N'') = k$.

Since $\mathrm{ne}(N') = k + 1$, from the way N'' is constructed it can readily be seen that $\mathrm{ne}(N'') \geq k$. Suppose $\mathrm{ne}(N'') > k$. Since the ne-chains are preserved under rotation by 180°, in the discussion below we can assume, without loss of generality, that in $\mathcal{M}$ the shorter row $\mathcal{R}_s$ is above $\mathcal{R}_l$. This will allow us to avoid introducing too much notation, and instead use words such as "below", "above", etc. Let $\mathcal{S}$ be the intersection of $\mathcal{M}$ (and therefore $\mathcal{N}$) and the vertical strip determined by the row $\mathcal{R}_s$. Precisely, let

$$\mathcal{S} = \{(x,y) : (x,y) \text{ is a cell in } \mathcal{M} \text{ such that there is a cell } (x,y') \in \mathcal{R}_s\}.$$

Lemma 10. *Every $(k+1)$-chain in N'' contains one 1-cell in each of the rows $\mathcal{R}_s$ and $\mathcal{R}_l$. Thus, every such chain is contained in the strip $\mathcal{S}$.*

Proof. Since $\mathrm{ne}(N'') > k$, N'' contains a $(k+1)$-chain C. If none of the elements of C is in $\mathcal{R}_s \cup \mathcal{R}_l$, then C is a chain in M, which yields a contradiction with $\mathrm{ne}(M) = k$. If C contains a 1-cell c from $\mathcal{R}_l$ but not a cell from $\mathcal{R}_s$ then C is a $(k+1)$-chain in N' also, which means that c is a problem cell, which contradicts the fact that during the construction of N'' all problem cells in N' are replaced by 0s. If C contains a 1-cell c from $\mathcal{R}_s$ but not a cell from $\mathcal{R}_l$ then there are 2 possible cases: (i) c is a 1-cell in N'. In this case C is a $(k+1)$-chain in M as well. This contradicts $\mathrm{ne}(M) = k$. (ii) c is a 0-cell in N'. In this case the cell d above c in row $\mathcal{R}_l$ is a problem 1-cell and C is contained in the strip $\mathcal{S}$. Then $(C - \{c\}) \cup \{d\}$ is a $(k+1)$-chain in M. This again contradicts $\mathrm{ne}(M) = k$. ∎

Let y_1—c—b—x_1 be a $(k+1)$-chain in N'', where c and b are the 1-cells in $\mathcal{R}_s$ and $\mathcal{R}_l$ respectively, y_1 is the part of the chain which is southwest of c and x_1 is the part of the chain which is northeast of b. Note that, since b is a 1-cell in N'', it is also a 1-cell in N' but it is not a problem cell.

Lemma 11. *The cell c is a 0-cell in N' and therefore the cell a in row $\mathcal{R}_l$ directly above c is a problem cell.*

Proof. If c is a 1-cell in N' then y_1—c—b—x_1 is a $(k+1)$-chain in N' as well, which implies that b is a problem cell. But then b would be a 0-cell in N'' and cannot be a part of a $(k+1)$-chain in N''. ∎

Therefore, the cell a is a part of a $(k+1)$-chain y_2—a—x_2 in N'. Moreover, this chain is not contained in the strip $\mathcal{S}$. Otherwise, y_2—a—x_2 is a $(k+1)$-chain in M as well. We will show that this implies a contradiction, as stated in Lemma 12. Figure 4 illustrates the positions of the two chains.

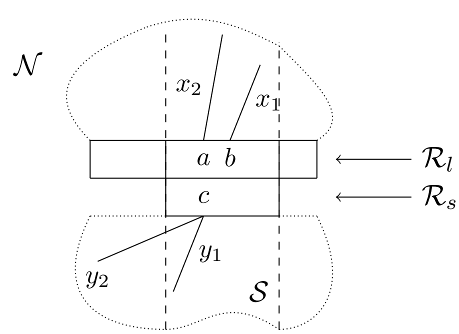

Figure 4: The $(k+1)$-chain y_1—c—b—x_1 in N'' is contained in the strip $\mathcal{S}$. The $(k+1)$-chain y_2—a—x_2 in N' is not contained in $\mathcal{S}$. In $N' : a = 1, b = 1, c = 0$, in $N'' : a = 0, b = 1, c = 1$.

Lemma 12. *Suppose, using the notation above,*

$$y_1—c—b—x_1 \text{ is a } (k+1)\text{-chain in } N'' \text{ contained in } \mathcal{S}, \tag{1}$$

$$y_2—a—x_2 \text{ is a } (k+1)\text{-chain in } N' \text{ not contained in } \mathcal{S}. \tag{2}$$

Then M has a $(k+1)$-chain or N' has a $(k+1)$-chain that contains b.

Proof. We will discuss cases according to the relative positions of x_1 and x_2. For the discussion, it is helpful to visualize a chain as a piece of string directed northeast connecting the centers of its 1-cells, so that it's piecewise linearly increasing. Then we will say that chains cross if the corresponding strings cross. Let $r_{\mathcal{M}}(d)$ denote the row in $\mathcal{M}$ that contains the cell d and define $r_{\mathcal{M}}(d_1) \leq r_{\mathcal{M}}(d_2)$ if the set of abscissas of the cells of row $r_{\mathcal{M}}(d_1)$ is contained in the set of abscissas of the cells of $r_{\mathcal{M}}(d_2)$. For any almost-moon polyomino, this defines a total order on its rows. Note that condition (2) implies that

$$\mathcal{R}_s \leq r_{\mathcal{N}}(d) \text{ for every } d \in x_2 \cup y_2. \tag{3}$$

Moreover,

$$C \text{ forms a } k\text{-chain if and only if every two cells in } C \text{ form a 2-chain,} \tag{4}$$

$$d_1 \in x_1 \text{ and } d_2 \in x_2 \text{ form a 2-chain if and only if they are in north-east direction,} \tag{5}$$

$$d_1 \in y_1 \text{ and } d_2 \in x_2 \text{ such that } d_2 \text{ is weakly to the left of a cell in } x_1 \text{ form a 2-chain,} \tag{6}$$

$$d_1 \in y_2 \text{ and } d_2 \in x_1 \text{ such that } d_2 \text{ is not higher than the highest point of } x_2 \text{ form a 2-chain,} \tag{7}$$

Suppose first that both x_1 and x_2 are nonempty.

(1) If the chains $b—x_1$ and $a—x_2$ cross, let P be the first intersection point. Let x_3 and x_4, respectively, be the parts of x_1 and x_2, respectively, that are strictly southwest of P. Then

- if $|x_4| > |x_3|$ then $y_1—a—x_4—(x_1 \backslash x_3)$ is at least a $(k+1)$-chain in M; the submatrix condition is satisfied because of (4), (5), and (6);
- if $|x_4| \leq |x_3|$ then $y_2—b—x_3—(x_2 \backslash x_4)$ is at least a $(k+1)$-chain in N'; the submatrix condition is satisfied because of (4), (5), and (7).

(2) If the chains $b—x_1$ and $a—x_2$ do not cross, there are 4 cases to be considered:

(a) x_2 ends weakly below and strictly to the left of x_1. Let x_3 be the part of x_1 that is in the rows weakly below the highest cell of x_2. Then
- if $|x_2| > |x_3|$ then $y_1—a—x_2—(x_1 \backslash x_3)$ is at least a $(k+1)$-chain in M; the submatrix condition is satisfied because of (4), (5), and (6);
- if $|x_2| \leq |x_3|$ then $y_2—b—x_3$ is at least a $(k+1)$-chain in N'; the submatrix condition is satisfied because of (4) and (7).

(b) x_2 ends strictly above and weakly to the left of x_1. Then
- if $|x_2| \leq |x_1|$ then $y_2—b—x_1$ is at least a $(k+1)$-chain in N'; the submatrix condition is satisfied because of (4) and (7);
- if $|x_2| > |x_1|$ then $y_1—a—x_2$ is at least a $(k+1)$-chain in M; the submatrix condition is satisfied because of (4) and (6).

(c) x_2 ends strictly above and to the right of x_1. Let x_4 be the part of x_2 that is in the columns weakly to the left of the end of x_2. Then
- if $|x_4| > |x_1|$ then y_1—a—x_4 is at least a $(k+1)$-chain in M;
the submatrix condition is satisfied because of (4) and (6);
- if $|x_4| \leq |x_1|$ then y_2—b—x_1—$(x_2 \backslash x_4)$ is at least a $(k+1)$-chain in N';
the submatrix condition is satisfied because of (4), (5), and (7).

(d) x_2 ends weakly below and weakly to the right of x_1. This case is not possible because then x_1 and x_2 must cross.

Finally, we consider the cases when one of x_1 and x_2 is empty.

(1) If $|x_2| = 0$ then y_2—b is a $(k+1)$-chain in N';
the submatrix condition is satisfied because of (3).

(2) If $|x_2| \neq 0$ but $|x_1| = 0$, let x_5 be the part of x_2 that is in the columns weakly to the left of b. Then

- if $|x_5| > 0$ then y_1—a—x_5 is at least a $(k+1)$-chain in M;
the submatrix condition is satisfied because of (4) and (6);

- if $|x_5| = 0$ then y_2—b—x_2 is a $(k+1)$-chain in N';
the submatrix condition is satisfied because of (3).

∎

To conclude, the construction of N'' and the preceding three lemmas imply the following property: For $M \in \mathcal{F}^{II}(M)$, the above constructed filling $N'' = \phi_{\mathcal{M},\mathcal{N}}(M)$ of $\mathcal{N}$ has the same column sums as M and $\mathrm{ne}(N'') = \mathrm{ne}(M)$.

We next prove some properties of the transformation of the, so far, partially defined map $\phi_{\mathcal{M},\mathcal{N}}$. These will be useful in extending the definition of $\phi_{\mathcal{M},\mathcal{N}}$ to $\mathcal{F}^{III}(\mathcal{M})$.

Lemma 13. *Let $M \in \mathcal{F}^{II}(\mathcal{M})$ with $\mathrm{ne}(M) = k$. Let $N'' = \phi_{\mathcal{M},\mathcal{N}}(M)$ be defined as in Case II. Then*

$$\mathrm{ne}(f_{\mathcal{N},\mathcal{M}}(N'')) = \mathrm{ne}(N'') + 1 = k + 1.$$

In other words, if $M \in \mathcal{F}^{II}(\mathcal{M})$ then $\phi_{\mathcal{M},\mathcal{N}}(M) \in \mathcal{F}^{II}(\mathcal{N})$. Moreover,

$$\phi_{\mathcal{N},\mathcal{M}}(\phi_{\mathcal{M},\mathcal{N}}(M)) = M.$$

Consequently, $\phi_{\mathcal{M},\mathcal{N}}$ restricted on $\mathcal{F}^{II}(\mathcal{M})$ is a bijection onto $\mathcal{F}^{II}(\mathcal{N})$.

Proof. Suppose A is the set of 1-cells in row $\mathcal{R}_l$ of $N' = f(M)$ and A_0 is the subset of A containing the problem cells. Let also B be the set of 1-cells in $\mathcal{R}_s$ of N' and B_0 the set of cells in $\mathcal{R}_s$ that share a horizontal edge with a cell inA_0. By construction, the set of 1-cells in row $\mathcal{R}_l$ of $N'' = \phi_{\mathcal{M},\mathcal{N}}(M)$ is $A_1 = A \backslash A_0$ and the set of 1-cells in row $\mathcal{R}_s$ is $B_1 = B \cup B_0$.

We now consider $M' = f_{\mathcal{N},\mathcal{M}}(N'') \in \mathcal{F}(\mathcal{M})$. Let $a \in A_0$ be a problem cell in N'. Then, by definition, a is a part of a $(k+1)$-chain y—a—x in N' and the cells in the chain are all contained in the vertical strip determined by row $\mathcal{R}_l$. Let $b \in B_0$ be the neighbor cell of a. Then y—b—x forms a $(k+1)$ chain in M'. Since M' contains a $(k+1)$ chain, Lemma 7 implies that $\mathrm{ne}(M') = k+1$.

The preceding argument also shows that the 1-cells in B_0 are problem cells in M' and it is not difficult to see that these are the only problem cells. Therefore, to construct $\phi_{\mathcal{N},\mathcal{M}}(N'') \in \mathcal{F}(\mathcal{M})$, we replace the 1-cells in B_0 by 0s and the neighboring 0-cells in A_0 by 1s, which yields the initial filling M. ∎

Lemma 14. *Let $M \in \mathcal{F}^{III}(\mathcal{M})$. Then $\phi_{\mathcal{M},\mathcal{N}}(M)$ is well defined, its column sums are the same as M's, and* $\mathrm{ne}(\phi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M)$. *Moreover, $\phi_{\mathcal{M},\mathcal{N}}(M) \in \mathcal{F}^{III}(\mathcal{N})$ and $\phi_{\mathcal{N},\mathcal{M}}(\phi_{\mathcal{M},\mathcal{N}}(M)) = M$. Consequently, $\phi_{\mathcal{M},\mathcal{N}}$ restricted on $\mathcal{F}^{III}(\mathcal{M})$ is a bijection onto $\mathcal{F}^{III}(\mathcal{N})$.*

Proof. To see that $\phi_{\mathcal{M},\mathcal{N}}(M)$ is well defined, let $N' = f_{\mathcal{M},\mathcal{N}}(M)$. Then $f_{\mathcal{N},\mathcal{M}}(N') = M$ and $N' \in \mathcal{F}^{II}(\mathcal{N})$ because $\mathrm{ne}(f_{\mathcal{N},\mathcal{M}}(N')) = \mathrm{ne}(N') + 1$. Therefore, $\phi_{\mathcal{N},\mathcal{M}}(N')$ is well defined and so is $\phi_{\mathcal{M},\mathcal{N}}(M)$. Additionally, Lemma 13 implies that $\phi_{\mathcal{N},\mathcal{M}}(N') \in \mathcal{F}^{II}(\mathcal{M})$. Therefore,

$$\mathrm{ne}(\phi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(\phi_{\mathcal{N},\mathcal{M}}(f_{\mathcal{M},\mathcal{N}}(M))) + 1 = \mathrm{ne}(f_{\mathcal{M},\mathcal{N}}(M)) + 1 = \mathrm{ne}(M).$$

It is clear that the column sums of M and $\phi_{\mathcal{M},\mathcal{N}}(M)$ are equal. Finally, $\phi_{\mathcal{N},\mathcal{M}}(\phi_{\mathcal{M},\mathcal{N}}(M)) = M$ follows from Lemma 6 and Lemma 13. ∎

4 Maximal increasing sequences in fillings with restricted row sum

In this section we restrict to fillings with at most one 1 in each row and prove Theorem 3. Explicitly, let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes that can be obtained from each other by an interchange of two adjacent rows. Assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. Let $\mathbf{r} \in \{0,1\}^*$ and $\mathbf{c} \in \mathbf{N}^*$, we shall construct a bijection $\psi_{\mathcal{M},\mathcal{N}}$ from $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ to $\mathcal{F}(\mathcal{M}, \mathbf{r}', \mathbf{c})$ that preserves the size of the largest ne-chains, where $\mathbf{r}'$ is obtained from $\mathbf{r}$ by exchanging the entries corresponding to the two swapped rows.

If the two swapped rows are of equal length, then $\mathcal{M} = \mathcal{N}$ and we can simply take $\psi_{\mathcal{M},\mathcal{N}}$ to be the identity map. In the following, we assume that the two swapped rows are $\mathcal{R}_s$ and $\mathcal{R}_l$, where the length of $\mathcal{R}_s$ is smaller than that of $\mathcal{R}_l$. We keep the notations as in the previous section. For any filling M let $\alpha, \beta, \gamma, \delta$ be as defined in Figure 3. Note that for $M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$, there is at most one 1 in α, as well as in the union of β, γ and δ. Define the *filling coupled with* M to be the filling M' of $\mathcal{M}$ which is obtained from M by exchanging the fillings α and β. Let N (resp. N') be obtained from M (resp. M') by swapping the rows $\mathcal{R}_s$ and $\mathcal{R}_l$ together with their fillings. In other words, $N = f_{\mathcal{M},\mathcal{N}}(M')$ and $N' = f_{\mathcal{M},\mathcal{N}}(M)$. Clearly N and N' are coupled fillings.

We need the following lemma, which is the crucial observation for the construction of $\psi_{\mathcal{M},\mathcal{N}}$.

Lemma 15. *Let (M, M') be a pair of coupled fillings in $\mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c})$, and*

$$(N, N') = (f_{\mathcal{M},\mathcal{N}}(M'), f_{\mathcal{M},\mathcal{N}}(M))$$

be fillings of $\mathcal{N}$. Then $\mathrm{ne}(M) = \mathrm{ne}(N)$ *or* $\mathrm{ne}(M) = \mathrm{ne}(N')$.

We postpone the proof of Lemma 15 and explain how it helps us construct the bijection $\psi_{\mathcal{M},\mathcal{N}}$. Lemma 15 implies that

$$\{\mathrm{ne}(M), \mathrm{ne}(M')\} = \{\mathrm{ne}(N), \mathrm{ne}(N')\} \tag{8}$$

as multisets. To see this, note that if $\mathrm{ne}(N) = \mathrm{ne}(N') = k$, then applying Lemma 15 to both M and M' yields $\mathrm{ne}(M) = \mathrm{ne}(M') = k$. Otherwise, if $\mathrm{ne}(N) \neq \mathrm{ne}(N')$, then applying Lemma 15 to N, N' yields that one of $\mathrm{ne}(M), \mathrm{ne}(M')$ equals $\mathrm{ne}(N)$, and the other equals $\mathrm{ne}(N')$. Equation (8) allows us to construct an ne-preserving bijection between the pairs of coupled fillings $\{M, M'\}$ and $\{N, N'\}$. Combining the bijections of all the coupled fillings we get a desired bijection. Explicitly, we can describe the map $\psi_{\mathcal{M},\mathcal{N}}$ as follows:

Let M be a filling in $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$.

1. If either α or β has no 1, then let $\psi_{\mathcal{M},\mathcal{N}}(M) = N$, the filling of $\mathcal{N}$ obtained by swapping the two rows together with their fillings.
2. If both α and β contain a 1 in the same column, then again let $\psi_{\mathcal{M},\mathcal{N}}(M) = N$.
3. If each of α and β contain a unique 1 in a distinct column,

$$\psi_{\mathcal{M},\mathcal{N}}(M) = \begin{cases} N & \text{if } \mathrm{ne}(M) = \mathrm{ne}(N) \\ N' & \text{if } \mathrm{ne}(M) \neq \mathrm{ne}(N). \end{cases}$$

It is clear that the map $\psi_{\mathcal{M},\mathcal{N}}$ is well-defined and preserving the statistic ne. In addition, $\psi_{\mathcal{M},\mathcal{N}}$ and $\psi_{\mathcal{N},\mathcal{M}}$ are inverse to each other.

Proof of Lemma 15. If either α or β contains no 1, or they both contain a 1 in the same column, then swapping the two rows with their fillings will not change any ne-chains, and hence $\mathrm{ne}(M) = \mathrm{ne}(N)$. In the following we assume that each of α and β contains a unique 1-cell in a distinct column, and there is no 1 in γ or δ. Furthermore we assume that in $\mathcal{M}$, the row $\mathcal{R}_s$ is above the row $\mathcal{R}_l$. The opposite case can be treated similarly.

We consider the relative positions of the two 1-cells in R_s and $\mathcal{R}_l$. See Figure 5 for an illustration.

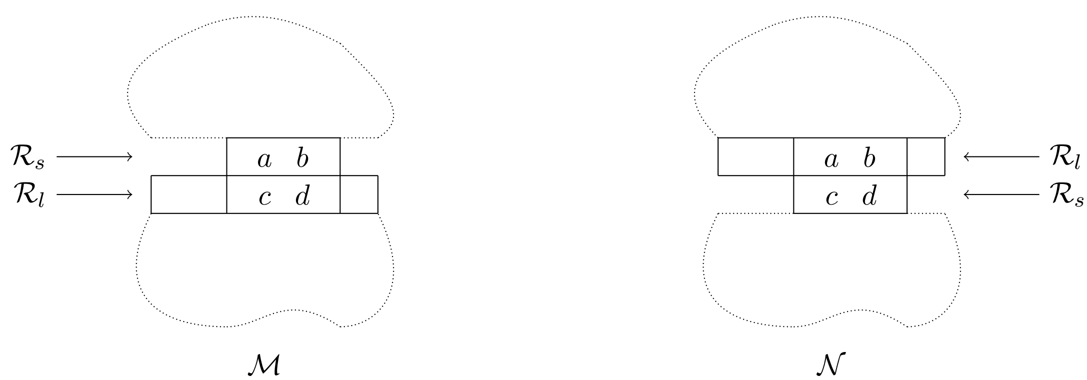

Figure 5: Positions of the 1-cells in the coupled fillings.

Case I. In the filling M the two 1-cells in rows $\mathcal{R}_s$ and $\mathcal{R}_l$ form an ne-chain of size 2. That is, the 1-cells of $\mathcal{M}$ in rows $\mathcal{R}_s$ and $\mathcal{R}_l$ are b and c as in the left figure of Figure 5 . Then N is the filling of $\mathcal{N}$ in which the only 1-cells in rows R_s and R_l are a and d, as in the right figure of Figure 5. Clearly $\mathrm{ne}(N) \leq \mathrm{ne}(M)$ as every ne-chain of N is still an ne-chain of M.

Assume $\mathrm{ne}(M) = k$. If $\mathrm{ne}(N) \neq \mathrm{ne}(M)$ then $\mathrm{ne}(N) = k - 1$, and every ne-chain of length k in M contains both b and c. In particular, N', the filling coupled with N which contains b and c in $\mathcal{N}$, has ne-chains of length k. It follows that $\mathrm{ne}(N') \geq k$.

Comparing fillings N and N'. Let C be an l-chain in N'.

(i) If C contains no 1-cells from $\mathcal{R}_s \cup \mathcal{R}_l$ then C is an l-chain in N.

(ii) If C contains the 1-cell c, then $C \cup \{a\} - \{c\}$ is an l-chain in N.

(iii) If C contains the 1-cell b, then $C - \{b\}$ is an $(l-1)$-chain in N.

(iv) If C contains both b and c, then $C \cup \{a\} - \{b, c\}$ is an $(l-1)$-chain in N.

Therefore $\mathrm{ne}(N) \geq \mathrm{ne}(N') - 1$ and hence $\mathrm{ne}(N') \leq \mathrm{ne}(N) + 1 \leq k$. Thus we must have $\mathrm{ne}(N') = k = \mathrm{ne}(M)$.

Case II. In the filling M the two 1-cells in rows $\mathcal{R}_s$ and $\mathcal{R}_l$ do not form an ne-chain of size 2. Then M has 1-cells a, d in $\mathcal{M}$, N has 1-cells b, c and N' has 1-cells a, d in $\mathcal{N}$. Similarly as in Case 1 we have

$$\mathrm{ne}(M) \leq \mathrm{ne}(N) \leq \mathrm{ne}(M) + 1$$
$$\mathrm{ne}(N) - 1 \leq \mathrm{ne}(N').$$

Using Lemma 7 we also have $|\mathrm{ne}(M) - \mathrm{ne}(N')| \leq 1$. Assume that $\mathrm{ne}(M) = k$ is not equal to $\mathrm{ne}(N)$ or $\mathrm{ne}(N')$. Combining these three inequalities, we derive that $\mathrm{ne}(N) = \mathrm{ne}(N') = k+1$. In addition,

(1) Every $(k+1)$-chain in N contains both b and c, and hence lies in the strip $\mathcal{S}$, where $\mathcal{S}$ is the union of columns that intersect the row $\mathcal{R}_s$, as defined in previous section.

(2) Every $(k+1)$-chain in N' contains a, and is not lying inside $\mathcal{S}$.

This is exactly the situation described in Lemma 12. Comparing with Figure 4 and applying Lemma 12, also noting that the filling N' does not have b as a 1-cell, we conclude that M has an ne-chain of length $k+1$. This is a contradiction. ∎

Although Lemma 12 plays an important role in the proofs of both Theorem 1 and 3, we remark that the map $\psi_{\mathcal{M},\mathcal{N}}$ is different than the map $\phi_{\mathcal{M},\mathcal{N}}$ restricted to the set $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$. For one thing, $\phi_{\mathcal{M},\mathcal{N}}$ does not always preserve the row sum, hence not necessarily maps fillings of $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ to $\mathcal{F}(\mathcal{N}, \mathbf{r}', \mathbf{c})$ when $\mathbf{r} \in \{0,1\}^*$.

5 Concluding remarks

We conclude this paper with some comments and counterexamples to a few seemingly natural generalizations of Theorem 1 and 3.

- *Symmetry of* (ne, se) *for fillings with* $\mathbf{r} \in \{0,1\}^*$.

  A *southeast* chain, or shortly *se-chain*, of size k in a 01-filling M is a set of k cells

$$\{(i_1, j_k), (i_2, j_{k-1}), \ldots, (i_k, j_1)\}$$

with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ filled with 1s such that the $k \times k$ submatrix

$$\mathcal{G} = \{(i_r, j_s) : 1 \leq r \leq k, 1 \leq s \leq k\}$$

is contained in the polyomino. We denote by se(M) the size of the largest se-chains of M. By the symmetry of ne and se we have that Lemma 15 also holds for se.

It is known that in 01-fillings of a Ferrers shape with fixed row sum and column sum in $\mathbf{N}$, the pair $(\mathrm{ne}, \mathrm{se})$ may not distribute symmetrically (e.g. see [16, 6]). On the other hand, when both $\mathbf{r}, \mathbf{c} \in \{0, 1\}$, $(\mathrm{ne}, \mathrm{se})$ does have a symmetric joint distribution. This was proved for Ferrers shapes by Krattenthaler [14] and for moon polyominoes by Rubey [16].

The above results raised the question whether for almost-moon polyominoes the pair of statistics $(\mathrm{ne}, \mathrm{se})$ has a symmetric joint distribution when one or both of $\mathbf{r}, \mathbf{c}$ are in $\{0, 1\}$, and whether the distribution of $(\mathrm{ne}, \mathrm{se})$ is unchanged when one swaps two adjacent rows.

Surprisingly, the answers are all negative. In the following we give two set of counterexamples. The first one is for the case that $\mathbf{r} \in \{0, 1\}$ but $\mathbf{c} \in \mathbf{N}$. The involved polyominoes are of small sizes, and we can list all the fillings explicitly. The second is for the case when both $\mathbf{r}$ and $\mathbf{c}$ are in $\{0, 1\}$. We found the counterexample by running a computer program, and we will just describe the results without listing all the details.

Example 16. *Figures 6 and 7 list all the 01-fillings of three polyominoes, where the polyominoes in Figure 7 are obtained from the moon polyomino in Figure 6 by moving down the first row. In all the fillings we require that* $\mathbf{r} = (1, 1, 1, 1)$ *and* $\mathbf{c} = (2, 1, 1)$*. The data* $(\mathrm{ne}, \mathrm{se})$ *is given under each filling.*

Figure 6 shows that even for moon polyomino with $\mathbf{r} \in \{0, 1\}$ but $\mathbf{c} \in \mathbb{N}$, the distribution of $(\mathrm{ne}, \mathrm{se})$ is not necessarily symmetric. Figure 7 gives an example that the distribution of the pair $(\mathrm{ne}, \mathrm{se})$ is not preserved when two adjacent rows are swapped in an almost-moon polyomino. Note that the first two fillings (M, M') in Figure 7 are coupled fillings, whose corresponding coupled fillings are the first two fillings (N, N') in the second row of Figure 7. For these two pairs Lemma 15 does not hold for $(\mathrm{ne}, \mathrm{se})$, i.e.,

$$\{(\mathrm{ne}(M), \mathrm{se}(M)), (\mathrm{ne}(M'), \mathrm{se}(M')\} \neq \{(\mathrm{ne}(N), \mathrm{se}(N)), (\mathrm{ne}(N'), \mathrm{se}(N'))\}.$$

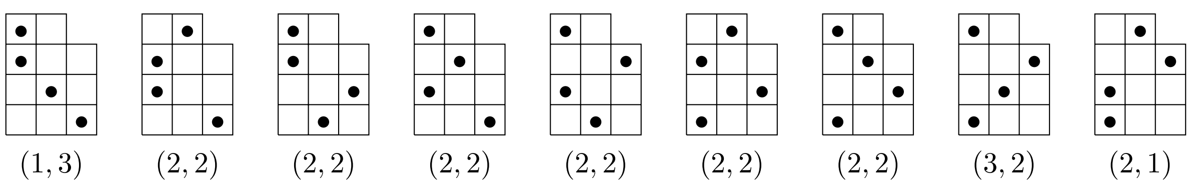

Figure 6: Fillings of a moon polyomino with $\mathbf{r} = (1, 1, 1, 1)$ and $\mathbf{c} = (2, 1, 1)$.

Example 17. *Our second example is restricted to 01-fillings where each row as well as each column has exactly one 1. We say that such fillings are* restricted*. Let* $\mathcal{M}_1$*,* $\mathcal{M}_2$ *and* $\mathcal{M}_3$ *be the polyominoes whose cells are given by*

$$\mathcal{M}_1 = \{(i, j) : \ 1 \le i \le 6, \ 1 \le j \le 6\} - \{(1, 6)\}.$$
$$\mathcal{M}_2 = \{(i, j) : \ 1 \le i \le 6, \ 1 \le j \le 6\} - \{(1, 5)\}.$$
$$\mathcal{M}_3 = \{(i, j) : \ 1 \le i \le 6, \ 1 \le j \le 6\} - \{(1, 4)\}.$$

Each polyomino has 600 restricted fillings. Let $G_1(x, y) = \sum_M x^{\mathrm{ne}(M)} y^{\mathrm{se}(M)}$ *be the joint distribution of* $(\mathrm{ne}, \mathrm{se})$ *over restricted fillings of* $\mathcal{M}_1$*. Similarly define* $G_2(x, y)$ *and* $G_3(x, y)$ *for restricted fillings*

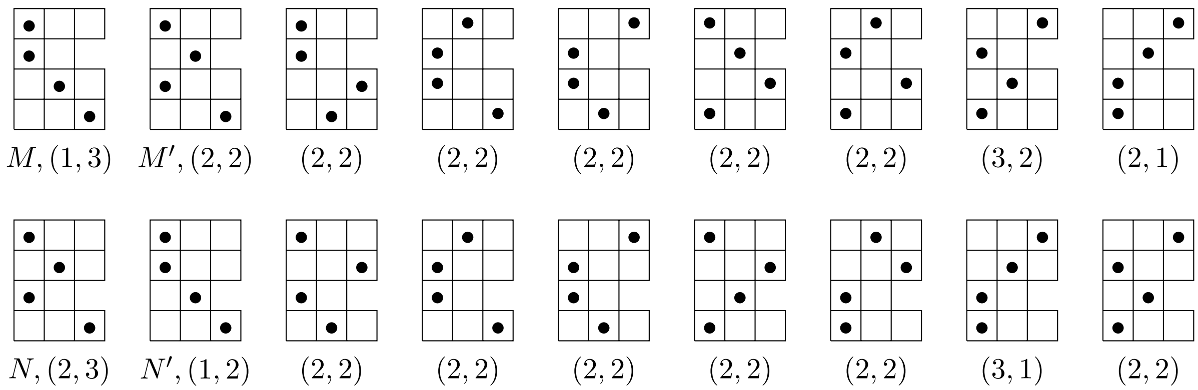

Figure 7: Fillings of almost-moon polyominoes. The first row is for an almost-moon polyomino $\mathcal{M}$, and the second row is for $\mathcal{N}$ which differ from $\mathcal{M}$ by swapping the second and the third rows.

in $\mathcal{M}_2$ and $\mathcal{M}_3$. With the help of a computer program we obtained

$$\begin{aligned} G_1(x,y) = G_2(x,y) = xy^5 + x^5y + 72(x^2y^4 + x^4y^2) + 48(x^3y^4 + x^4y^3) \\ +50(x^2y^3 + x^3y^2) + 8(x^2y^5 + x^5y^2) + 242x^3y^3 \end{aligned} \tag{9}$$

and

$$\begin{aligned} G_3(x,y) = xy^5 + x^5y + 72x^2y^4 + 73x^4y^2 + 48x^3y^4 + 47x^4y^3 \\ +50x^2y^3 + 49x^3y^2 + 8x^2y^5 + 8x^5y^2 + 243x^3y^3. \end{aligned} \tag{10}$$

Equation (9) is symmetric with respect to x, y, which is expected for $\mathcal{M}_1$ since it is a moon polyomino. Equation (10) shows that the joint distribution of (ne, se) over almost-moon polyominoes is not necessarily symmetric, even if we require that every row and every column has exactly one 1. The difference between the two equations implies that the distribution of (ne, se) may not be preserved when two adjacent rows are swapped.

- *Coupled fillings with* $\mathbf{r} \in \mathbb{N}^*$.

Another question is whether we can extend the idea of *coupling* in Theorem 3 to construct a bijection for Theorem 1. The following example shows that the direct application does not work.

Example 18. *Consider fillings in $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$ where a row may have multiple 1s. For a filling M of $\mathcal{M}$, we couple with it the filling M' obtained from M by swapping the fillings α and β, just as what we did in Section 4. Again let $N = f_{\mathcal{M},\mathcal{N}}(M')$ and $N' = f_{\mathcal{M},\mathcal{N}}(M)$. In the fillings shown in Figure 8, $\mathrm{ne}(M) = 3$, $\mathrm{ne}(M') = 4$ while $\mathrm{ne}(N) = \mathrm{ne}(N') = 4$. Lemma 15 does not hold for this case.*

It is still open whether Theorem 3 holds for the family of fillings that $\mathbf{r} \in \mathbb{N}^*$ but $\mathbf{c} \in \{0,1\}^*$ We point out that Lemma 15 does not hold in this case either. Given a filling M with multiple 1-cells in a row, the natural way to define the coupling with the same row sum is to let M' be obtained from M by keeping the empty columns of $\alpha \cup \beta$ and reversing the fillings in the remaining columns of $\alpha \cup \beta$. Then (N, N') are obtained from (M, M') by exchanging the two rows $\mathcal{R}_s$ and R_l with their fillings.

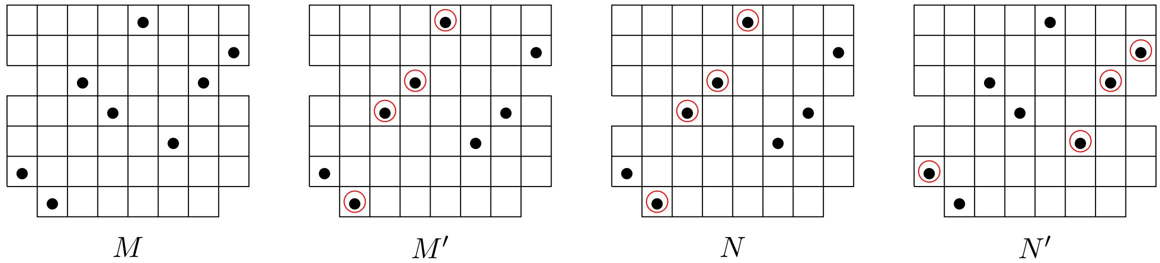

Figure 8: Coupled fillings in $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ with $\mathbf{r} \in \mathbb{N}^*$ and $\mathbf{r} \in \{0,1\}^*$. The circled dots form 4-chains in the polyominoes.

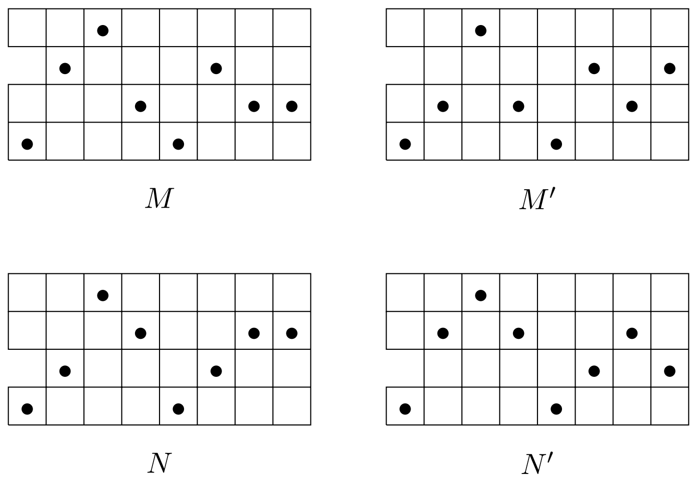

Figure 9: Coupled fillings in $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ with $\mathbf{r} \in \mathbb{N}^*$ and $\mathbf{c} \in \{0,1\}^*$.

Example 19. *The fillings in Figure 9 gives an example where* $\mathrm{ne}(M) = 2, \mathrm{ne}(M') = 3$ *while* $\mathrm{ne}(N) = \mathrm{ne}(N') = 3$.

- *Statistic* ne *in fillings of general polyominoes*

It is natural to ask whether Theorem 1 or Theorem 3 can be extended to a more general family of polyominoes, for example, to *layer polyominoes*, which are polyominoes that are row-convex and row-intersection-free. Layer polynomials were first studies in [15] which inspired the current work. The following example from [15] shows that if there are several exceptional rows in the polyomino, then the distribution of $\mathrm{ne}(M)$ may not be the same after a swap of two adjacent rows.

Example 20. *Consider the polyominoes given in Figure 10. The right one is almost-moon but the left one is not. Let* $G(x, \mathcal{F}) = \sum_{M \in \mathcal{F}} x^{\mathrm{ne}(M)}$ *be the generating function for the statistic* ne *over the fillings in a set* $\mathcal{F}$. *First consider 01-fillings where every row and every column has exactly one* 1. *We obtained the following generating functions. For the left polyomino* $\mathcal{M}_1$,

$$G(x, \mathcal{F}(\mathcal{M}_1, \mathbf{r}, \mathbf{c})) = x + 37x^2 + 31x^3 + 3x^4,$$

while for the right polyomino $\mathcal{M}_2$,

$$G(x, \mathcal{F}(\mathcal{M}_2, \mathbf{r}, \mathbf{c})) = x + 36x^2 + 32x^3 + 3x^4,$$

where $\mathbf{r} = \mathbf{c} = (1,1,1,1,1)$. *This provides a counterexample of Theorem 3 for general polyominoes.*

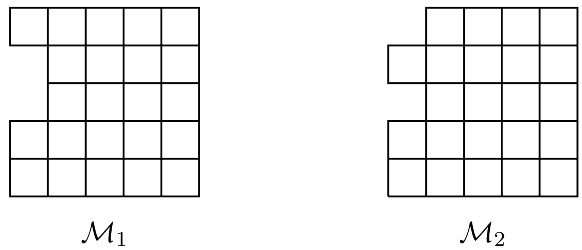

Figure 10: Two general polyominoes related by an interchange of two adjacent rows.

The above example also implies that Theorem 1 can not hold for general polyominoes. Namely, take the same two polyominoes and consider all the fillings in which each column has exactly one 1, but there is no constraint on the row. Note that

- Empty rows do not affect the statistic ne and can be ignored.
- Any sub-polyomino of $\mathcal{M}_1$ or $\mathcal{M}_2$ containing three rows is an almost-moon polyomino. By Theorem 1 rearranging rows for such three-row polyominoes does not change the distribution of $\mathrm{ne}(M)$ over $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$.

Now let $\mathbf{c} = (1,1,1,1,1)$ we have

1. Either $G(\mathrm{ne}, \mathcal{F}(\mathcal{M}_1, *, \mathbf{c})) \neq G(\mathrm{ne}, \mathcal{F}(\mathcal{M}_2, *, \mathbf{c}))$ and we have the desired counterexample, or
2. $G(\mathrm{ne}, \mathcal{F}(\mathcal{M}_1, *, \mathbf{c})) = G(\mathrm{ne}, \mathcal{F}(\mathcal{M}_2, *, \mathbf{c}))$. But then the example from the previous paragraph implies that the distribution of ne over 01-fillings in $\mathcal{F}(\mathcal{M}_1, *, \mathbf{c})$ and $\mathcal{F}(\mathcal{M}_2, *, \mathbf{c})$ with empty rows are different. Note that the set of fillings of a polyomino $\mathcal{M}$ with empty rows can be obtained as the union of set $\mathcal{F}_i(\mathcal{M}, \mathbf{r}, \mathbf{c})$, which consists of fillings in which the i-th row is empty. An application of the inclusion-exclusion principle implies that there is a sub-polyomino $\mathcal{N}_1$ of $\mathcal{M}_1$ consisting of 4 rows such that $G(\mathrm{ne}, \mathcal{F}(\mathcal{N}_1, *, \mathbf{c})) \neq G(\mathrm{ne}, \mathcal{F}(\mathcal{N}_2, *, \mathbf{c}))$ where $\mathcal{N}_2$ is the sub-polyomino of $\mathcal{M}_2$ consisting of the same four rows as in $\mathcal{N}_1$.

- *A few open questions*

Finally, we list a few open questions related to the work in this paper.

Q1. It was shown in [17] that maximal 01-fillings of moon polyominoes with restricted chain lengths can be identified with certain rd-graphs, also known as pipe dreams. Is there an extension of pipe dreams to almost-moon polyominoes? If so, is there a relation to the conjecture of Rubey [17, Conj. 2.8] concerning the lattice structure of the poset of chute moves?

Q2. It would be interesting to see whether there is a relationship between the Edelman-Greene bijection as used in [18, 17] and the map on moon polyominoes induced by the $\phi_{\mathcal{M},\mathcal{N}}$'s. The analogous question for the Robinson-Schensted map as used in [14] and the BWX map was answered by Bloom and Saracino [2].

Q3. Is there a "nice" bijection that reverses the sequence of the rows of the polyomino?

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