Avoiding Monotone Chains in Fillings of Layer Polyominoes

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Abstract

In this paper we give simple bijective proofs that the number of fillings of layer polyominoes with no northeast chains is the same as the number with no southeast chains. We consider 01-fillings and $\mathbb{N}$-fillings and prove the results for both strong chains where the smallest rectangle containing the chain is also in the polyomino, and for regular chains where only the corners of the smallest rectangle containing the chain are required to be in the polyomino.

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1 Introduction

Let G be a simple graph on $[n] := \{1, 2, \ldots, n\}$. A graph can be represented by its set of edges where the edge $\{i, j\}$ is written (i, j) if $i < j$.

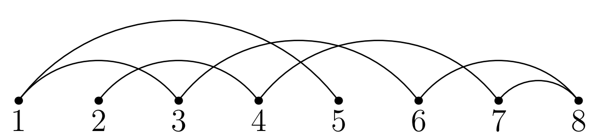


Figure 1: The graph on [7] with edges $\{(1,3),(1,5),(2,4),(3,6),(4,7),(6,8),(7,8)\}$.

A *crossing* in G (or *nesting*) is a pair of edges (i_1, j_1) and (i_2, j_2) so that $i_1 < i_2 < j_1 < j_2$ (resp., $i_1 < i_2 < j_2 < j_1$). If we draw the vertices of G in increasing order on a line and draw the arcs above the line (see Figure 1 for an example), crossings and nestings have a clear representation. The number of crossings (nestings) in a graph G is denoted $\text{cros}_2(G)$ (resp., $\text{nest}_2(G)$). A graph with no crossings (nestings, respectively) is called *noncrossing* (nonnesting, respectively).

The enumeration of noncrossing graphs, as well as other similar noncrossing configurations such as trees, forests, dissections, and partitions, have been studied in many research papers. Nonnesting configurations receive less attention. Nevertheless, it is observed that in many combinatorial structures, there is a symmetry between noncrossing and nonnesting configurations. Two well-known examples are set partitions and complete matchings. In particular, the following four structures:

noncrossing and nonnesting partitions on the set $[n]$, and noncrossing and nonnesting complete matchings of $[2n]$, are all counted by the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

See Stanley [13, Exercise 6.19] for a detailed list of Catalan structures.

De Sainte-Catherine [5] improved the above result on matchings by showing that the statistics nest_2 and cros_2 are equidistributed over all matchings of $[2n]$. Similar results have been found for set partitions, linked partitions and permutations (see e.g. [1, 2, 3, 7, 8, 10]).

More recently Kasraoui [6] generalized these results to 01-fillings of moon polyominoes, which is a set of connected square cells in $\mathbb{Z}^2$ that are convex and intersection-free. In the language of fillings of polyominoes, crossings and nestings of graphs correspond to northeast and southeast chains of length 2. Let $\mathrm{ne}(M)$ (resp. $\mathrm{se}(M)$) be the number of northeast (resp. southeast) chains of length 2 in a filling M. Kasraoui showed that the joint statistic $(\mathrm{ne}, \mathrm{se})$ is symmetric over 01-fillings of moon polyominoes with at most one non-zero entry per column. He further demonstrated that this joint distribution is invariant under permutations of the rows of the underlying polyomino as long as the permuted polyomino remains a moon polyomino. Phillipson, Yan and Yeh [9] extended these results to layer polyominoes, which are obtained from moon polyominoes by arbitrary permutations of the rows. It follows from [9] that when restricted to 01-fillings of layer polyominoes with at most one non-zero entry per column, the number of fillings with no northeast chains equals the number of fillings with no southeast chains.

It is known that even in Ferrers diagrams, the number of northeast and southeast chains of length 2 are not equidistributed over arbitrary 01-fillings, see for example, [6, Section 6]. On the other hand, there is a general result on fillings avoiding chains of length k. Considering fillings of a moon polyomino with given row sums but no restriction on column sums, Rubey [12] showed that the number of fillings with no northeast chains of length k is the same as those with no southeast chains of length k, for all positive integer k, and for both $\mathbb{N}$-fillings and 01-fillings. Rubey's results are proved by an adaption of *jeu de taquin* and promotion. A bijective proof for 01-fillings is given by Poznanovic and Yan [11] by introducing the notion of almost-moon polyominoes. In addition, for $\mathbb{N}$-fillings on Ferrers diagrams with given row and column sums, a bijective proof is given by de Mier [4]. All the proofs for the general case are quite technical.

In this paper we present simple bijective proofs which show the equality between the numbers of fillings avoiding northeast chains and those avoiding southeast chains. We consider both $\mathbb{N}$- and 01-fillings over the more general family of polyominoes—layer polyominoes. For a layer polyomino, there are two ways to extend the notion of 2-chains, the strong ones and the regular ones, whose definitions, as well as other necessary notations, are given in Section 2. Section 3 has a bijection between fillings of two layer polyominoes obtained from each other by swapping two adjacent rows. This bijection implies the equation for regular chains. Furthermore, we characterize $\mathbb{N}$-fillings with given row and column sums and no ne-chains. (See Section 2 for definitions.) Section 4 deals with strong chains. Using a framework with different operations, we prove three distinct results on the number of fillings with no strong chains, as well as on the distribution of strong chains.

2 Preliminaries

2.1 Polyominoes

A *polyomino* is a finite subset of $\mathbb{Z}^2$, where we regard an element of $\mathbb{Z}^2$ as a cell. A column of the polyomino is a set of cells along a vertical line, a row is the set of cells along a horizontal line. As convention we number rows top to bottom and columns left to right.

The polyomino is row (column) convex if the intersection with any horizontal (vertical) line is convex. It is intersection–free if for any two rows the column coordinates of one are contained in the column coordinates of the other. For example, the polyomino in Figure 2 is row–convex but neither column–convex nor intersection–free.

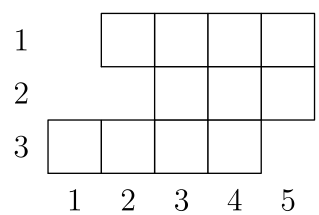


Figure 2: A row-convex polyomino, which is neither intersection–free nor column–convex.

Definition 2.1. A *layer polyomino* is a row–convex, intersection–free polyomino. A *moon polyomino* is a column–convex layer polyomino.

Figure 3 shows examples of both types of polyominoes.

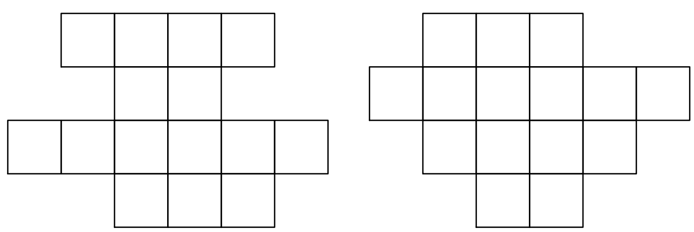


Figure 3: A layer and a moon polyomino.

2.2 Fillings and Chains

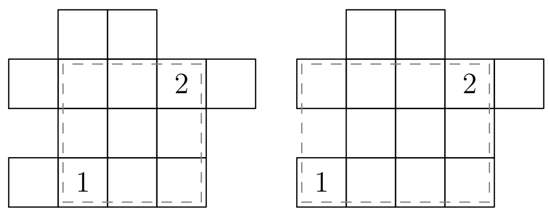
An $\mathbb{N}$-filling of a polyomino is an assignment of the non-negative integers to the cells of the polyomino. A 01-filling of a polyomino is an assignment of the numbers 0 and 1 to the cells of the polyomino. In figures zero entries in the polyomino are left empty.

The row sums of a filling of a polyomino is a vector $\vec{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$ so that r_i is the sum of the entries in the i^{th} row of the polyomino. Column sums are defined analogously and represented by a vector $\vec{c}$.

For a polyomino $\mathcal{P}$ with n rows and m columns, given vectors $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, we define $\mathcal{F}_{\mathbb{N}}(\mathcal{P}, \vec{r}, \vec{c})$ (resp. $\mathcal{F}_{01}(\mathcal{P}, \vec{r}, \vec{c})$) to be the collection of $\mathbb{N}$-fillings (resp. 01-fillings) of $\mathcal{P}$ with row sums $\vec{r}$ and column sums $\vec{c}$. In addition, we define $\mathcal{F}_{01}(\mathcal{P}, \vec{r})$ to be the collection of 01-fillings of $\mathcal{P}$ with row sums $\vec{r}$ and unrestricted column sums.

Definition 2.2. Let P be a filling of a polyomino $\mathcal{P}$. Two non-zero entries at (r_1, c_1) and (r_2, c_2) in P form a *regular northeast chain*, or ne–chain, if $r_2 < r_1$, $c_1 < c_2$, and the cells (r_1, c_2) and (r_2, c_1) are also contained in $\mathcal{P}$. Similarly a *regular southeast chain*, or se–chain, has non-zero entries at (r_1, c_2) and (r_2, c_1) while cells (r_1, c_1) and (r_2, c_2) are contained in $\mathcal{P}$.

Definition 2.3. A *strong northeast chain*, or $\text{ne}^{\square}$–chain, in a filling P of a polyomino $\mathcal{P}$, is a ne–chain so that the smallest rectangle containing the entries is contained in the polyomino. More precisely, the non-zero entries (r_1, c_1) and (r_2, c_2) in P form a $\text{ne}^{\square}$–chain if $r_2 < r_1$, $c_1 < c_2$, and all the cells in the set $\{(r, c) \mid r_2 < r < r_1 \text{ and } c_1 < c < c_2\}$ are contained in $\mathcal{P}$. We similarly define *strong southeast chains*, or $\text{se}^{\square}$–chains.



(a) Both a ne–chain and a $\text{ne}^{\square}$–chain

(b) A ne–chain, but not a $\text{ne}^{\square}$–chain

Figure 4: Chains in a layer polyomino.

Figure 4 gives examples of the previous definitions. Note that strong and regular chains coincide on moon polyominoes. As we will see that regular chains are a more natural generalization from chains of moon polyominoes as Kasraoui's [6] result on permuting rows in moon polyominoes extends to regular chains.

Definition 2.4. A filling of a polyomino is said to *avoid* a chain if there are no occurrences of that chain in the polyomino. For a polyomino $\mathcal{P}$ and a collection of fillings of $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$, let

$$\text{Av}(\text{ne}, \mathcal{F}(\mathcal{P})) = \{P \in \mathcal{F}(\mathcal{P}) \mid P \text{ has no ne–chains}\}.$$

We define sets $\text{Av}(\text{se}, \mathcal{F}(\mathcal{P}))$, $\text{Av}(\text{ne}^{\square}, \mathcal{F}(\mathcal{P}))$, and $\text{Av}(\text{se}^{\square}, \mathcal{F}(\mathcal{P}))$ similarly. Also we define $\text{av}(\text{ne}, \mathcal{F}(\mathcal{P}))$ to be the cardinality of the set $\text{Av}(\text{ne}, \mathcal{F}(\mathcal{P}))$ and similarly $\text{av}(\text{se}, \mathcal{F}(\mathcal{P}))$, $\text{av}(\text{ne}^{\square}, \mathcal{F}(\mathcal{P}))$, and $\text{av}(\text{se}^{\square}, \mathcal{F}(\mathcal{P}))$.

3 Regular Chains in Layer Polyominoes

In this section we prove several results about regular chains in fillings of layer polyominoes. For both 01- and $\mathbb{N}$- fillings we prove that an arbitrary permutation of the rows preserves the numbers of fillings with either no ne–chains or no se–chains. Further, for $\mathbb{N}$-fillings we give necessary and sufficient conditions with which $\mathbb{N}$–fillings exist with given row and column sums, and prove that under those conditions, the filling with no ne-chains (reps, se-chains) is unique.

In [9], the authors showed in fillings of layer polyominoes with at most 1 non-zero entry per column, the joint distribution (ne, se) is unaffected by an arbitrary permutation of rows. However, if we consider 01-fillings with fixed row and column sums in $\mathbb{N}^*$, then the symmetry of (ne, se) may not hold. Moreover, for a layer polyomino $\mathcal{L}$ and $\sigma \in S_n$ it is no longer true that $\text{av}(\text{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) =$

$\mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\sigma(\mathcal{L}), \sigma(\vec{r}), \vec{c}))$. For example for row sums $\vec{r} = (1, 2)$ and column sums $\vec{c} = (1, 2)$, there is only one possible filling of a 2×2 rectangle, Figure 5(a), and this filling has one ne–chain. On the other hand, transposing the rows gives $\sigma(\vec{r}) = (2, 1)$, but there is only one 01-filling with row sums $\sigma(\vec{r})$ and column sums $\vec{c}$, which is given in Figure 5(b) and has no ne–chains.

	1
1	1

(a) Row sums $(1, 2)$

1	1
	1

(b) Row sums $(2, 1)$

Figure 5: Fillings with fixed row sums and column sums

Our first result states that the number of fillings with no ne–chain is preserved under permutations of rows if we only fix row sums, but have no restrictions on column sums.

Theorem 3.1. *Let $\mathcal{L}$ be a layer polyomino with rows $R_1, \ldots, R_n$ from top to bottom and $\vec{r} \in \mathbb{N}^n$. For a permutation $\sigma \in \mathcal{S}_n$, let $\mathcal{L}' = \sigma(\mathcal{L})$ be the polyomino with rows $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$ and $\vec{r'} = \sigma(\vec{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. Let $\mathcal{F}$ be either* $\mathbb{N}$- *or* 01-*fillings. Then*

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}', \vec{r'})),$$

and

$$\mathrm{av}(\mathrm{se}, \mathcal{F}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{se}, \mathcal{F}(\mathcal{L}', \vec{r'})).$$

Proof. Any permutation can be obtained using a transposition of two adjacent rows, so we will show that permuting two consecutive rows preserves the number of fillings with no ne–chains. The case for se-chains is similar.

Let R_i and R_{i+1} be two consecutive rows, $L \in \mathrm{Av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}, \vec{r}))$ and let $\mathcal{R}$ be the largest rectangle contained in $R_i \cup R_{i+1}$. Construct L' from L by

1. exchanging R_i and R_{i+1} with their fillings,
2. fixing the empty columns of $\mathcal{R}$, and
3. reversing the filling of each row of $\mathcal{R}'$, where $\mathcal{R}'$ is the rectangle consisting of all the nonempty columns of $\mathcal{R}$.

Then $L' \in \mathrm{Av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}', \vec{r'}))$ as the above operations preserve fillings on $\mathcal{L} - \mathcal{R}$, the empty columns of $\mathcal{R}$, and changes the row sum from $\vec{r}$ to $\sigma(\vec{r})$. This guarantees that no new ne–chains are created involving cells outside of $\mathcal{R}$. Additionally $\mathcal{R}$ will still not have ne–chains as flipping both the rows and columns preserves this property. Figure 6 shows an example of this process, the cells are labeled for clarity and the rectangle $\mathcal{R}$ is boxed. In this example we flip rows 2 and 3, and shade the cells that we do not reverse. □

Corollary 3.2. *For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$, let $\mathcal{F}$ be either* 01- *or* $\mathbb{N}$- *fillings. Then*

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{se}, \mathcal{F}(\mathcal{L}, \vec{r})).$$

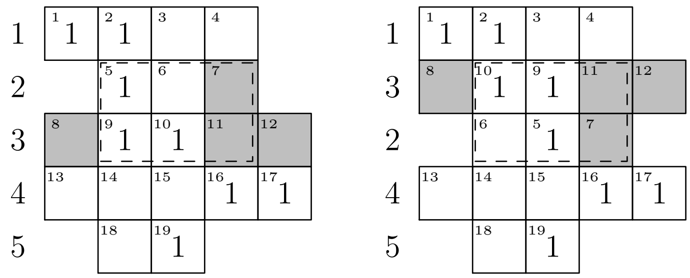


Figure 6: An example of the process in Theorem 3.1.

Proof. Let $\tilde{\mathcal{L}}$ be the polyomino obtained by reversing the rows of $\mathcal{L}$, that is, $\tilde{\mathcal{L}} = \sigma(\mathcal{L})$ where $\sigma = n \cdots 21$. For $L \in \mathcal{F}(\mathcal{L}, \vec{r})$, let $\tilde{L}$ be obtained from L by reversing the rows of L together with their fillings. Then $\tilde{L} \in \mathcal{F}(\tilde{\mathcal{L}}, \sigma(\vec{r}))$. It is clear that L has no se-chains if and only if $\tilde{L}$ has no ne-chains. Hence

$$\mathrm{av}(\mathrm{se}, \mathcal{F}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}(\tilde{\mathcal{L}}, \sigma(\vec{r}))) = \mathrm{av}(\mathrm{ne}, \mathcal{F}(\mathcal{L}, \vec{r})).$$

The last equation follows from Theorem 3.1. □

Next we give necessary and sufficient conditions when $\mathbb{N}$–fillings of layer polyominoes exist with given row and column sums. We also show that when such fillings do exist there is a unique one with no ne–chains. First we show the result for rectangles, then extend to Ferrers diagrams and finally to layer polyominoes by transforming $\mathbb{N}$-fillings of a layer polyomino to those of a Ferrers diagram with permutations of the rows and columns.

Lemma 3.3. *Let $\mathcal{R}$ be an $n \times m$ rectangle, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$. Then $\mathbb{N}$-fillings of $\mathcal{R}$ with row sums $\vec{r}$ and column sums $\vec{c}$ exist if and only if*

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} c_j. \tag{1}$$

Further, if $\vec{r}$ and $\vec{c}$ satisfy (1) then there is a unique filling of $\mathcal{R}$ with no ne*–chains (reps.* se*–chains).*

Proof. The necessity of (1) is clear. To prove sufficiency we use a greedy algorithm, implemented inductively. Let $\mathcal{R}$, $\vec{r}$ and $\vec{c}$ be as in the statement and satisfy (1). If $n = 1$ we can fill cell $(1, i)$ with c_i; similarly if $m = 1$ we fill cell $(i, 1)$ with r_i.

In general, we have three cases to consider: $r_1 = c_1$, $r_1 < c_1$, and $r_1 > c_1$; however, the last two cases are similar. If $r_1 = c_1$, fill cell $(1, 1)$ with r_1, cells $(1, i)$, $(j, 1)$ with 0 for $1 < i \leq m$ and $1 < j \leq n$, and reduce the problem to an $(n-1) \times (m-1)$ rectangle.

Assume $r_1 < c_1$. Then we fill cell $(1, 1)$ with r_1, fill cell $(1, i)$ with 0 for $1 < i \leq m$, and reduce the problem to an $(n-1) \times m$ rectangle with row sums $(r_2, \ldots, r_n)$ and column sums $(c_1 - r_1, c_2, \ldots, c_m)$. Continuing this process inductively produces a filling R with no ne–chains.

If f_{ij} is the entry in the (i, j) cell of the above constructed R, then each non-zero f_{ij} has the property that either $\sum_{\ell=1}^{i} f_{\ell j} = c_j$ or $\sum_{\ell=1}^{j} f_{i\ell} = r_i$.

Now we show that the filling R is the unique one with no ne-chains. Let R' be a different filling in $\mathcal{F}_{\mathbb{N}}(\mathcal{R}, \vec{r}, \vec{c})$ with entries f'_{ij} in cell (i, j). Find a cell i, j with minimal indices such that $f'_{ij} \neq f_{ij}$.

Then we must have $0 \leq f'_{ij} < f_{ij}$ and hence

$$\sum_{\ell=1}^{i} f'_{\ell j} < c_j \qquad \text{and} \qquad \sum_{\ell=1}^{j} f'_{i\ell} < r_i. \tag{2}$$

Therefore there exist non-zero entries $f_{kj} \neq 0$ with $i+1 \leq k \leq n$ and $f_{i\ell} \neq 0$ with $j+1 \leq \ell \leq m$, which means the entries f_{kj} and f_{il} form a ne–chain in R'. □

Now we extend this result to Ferrers diagrams. The conditions for a Ferrers diagram are slightly more complex than for rectangles.

Theorem 3.4. *Given a Ferrers diagram $\mathcal{T}$ with n rows and m columns, vectors $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, an $\mathbb{N}$-filling of $\mathcal{T}$, with row sums $\vec{r}$ and column sums $\vec{c}$, exists if and only if the following conditions hold:*

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} c_j, \tag{3}$$

$$\sum_{i \in S} r_i \leq \sum_{j:\exists i \in S((i,j)\in\mathcal{T})} c_j, \qquad \forall S \subseteq [n], \tag{4}$$

Further, if $\vec{r}$ and $\vec{c}$ satisfy (3) and (4) then the filling of $\mathcal{T}$ with no ne*–chains (*se*–chains) is unique.*

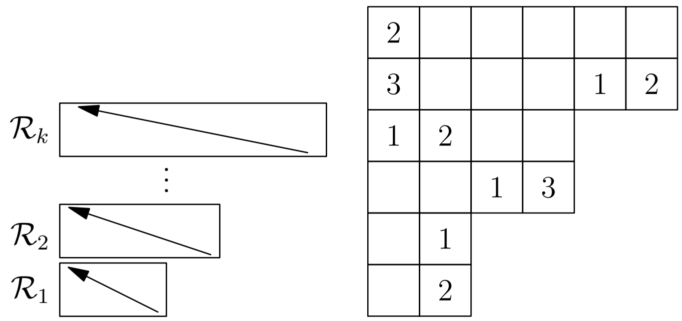
Proof. Partition the Ferrers diagram into a collection of rectangles $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$, where $\mathcal{R}_i$ is the union of the i^{th} shortest rows, see Figure 7(a). Starting with the rectangle $\mathcal{R}_1$, fill $\mathcal{R}_1$ using the greedy algorithm in the preceding proof from lower right to upper left, until all the cells of $\mathcal{R}_1$ are filled. By condition (4) this is possible, and in the resulting filling all the rows of $\mathcal{R}_1$ are saturated, i.e., the row sum of each row in $\mathcal{R}_1$ equals the desired row sum given by $\vec{r}$, and the column sums of $\mathcal{R}_1$ are no more than $\vec{c}$.

Subtract the column sums in $\mathcal{R}_1$ from the corresponding entries of $\vec{c}$ to get $\vec{c'}$. One checks that the conditions (3) and (4) still hold for the Ferrers diagram $\mathcal{T} \setminus \mathcal{R}_1$ with the row sums $\vec{r'}$ and column sums $\vec{c'}$, where $\vec{r'}$ is the restriction of $\vec{r}$ to the rows in $\mathcal{T} \setminus \mathcal{R}_1$. Then we continue inductively until we reach the last rectangle $\mathcal{R}_k$, which is filled and the row sums and columns sums are both saturate, by Lemma 3.3. Figure 7(b) shows an example with row sums $(2,6,3,4,1,2)$ and column sums $(6,5,1,3,1,2)$.

This filling has no ne–chains as if a column is saturated in $\mathcal{R}_i$, that column will remain empty in each subsequent $\mathcal{R}_j$, $j > i$. Additionally, the filling is unique by the same argument as in the proof of Lemma 3.3. □

Note that if the Ferrers diagram is aligned at the top and the left as in English notation, (e.g, Figure 7(b)), then condition (4) is equivalent to the following set of inequalities: for each i such that the row R_i of $\mathcal{T}$ is the top row of some rectangle $\mathcal{R}_j$,

$$\sum_{j=i}^{n} r_i \leq \sum_{j:(i,j)\in\mathcal{F}} c_j.$$



(a) A description of how to fill a Ferrer's shape with no ne-chains

(b) An example with no ne–chains

Figure 7

In addition, conditions (3) and (4) imply that

$$\forall T \subseteq [m], \quad \sum_{j \in T} c_j \leq \sum_{i: \exists j \in T((i,j) \in \mathcal{F})} r_i \tag{5}$$

Theorem 3.5. *Let $\mathcal{L}$ be a layer polyomino with rows $R_1, \ldots, R_n$ from top to bottom, m columns, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \mathbb{N}^m$. Let $\mathcal{L}' = \sigma(\mathcal{L})$ be the polyomino with rows $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$ and $\vec{r'} = \sigma(\vec{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. Then*

$$\text{av}(\text{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \text{av}(\text{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}', \vec{r'}, \vec{c}))$$

and

$$\text{av}(\text{se}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \text{av}(\text{se}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}', \vec{r'}, \vec{c})).$$

Proof. Proceeding in a manner similar to Theorem 3.1, we will show that we can permute any two adjacent rows while preserving the number of fillings with no ne–chains.

Let R_i and R_{i+1} be two consecutive rows, $L \in \text{Av}(\text{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ and $\mathcal{R}$ be the largest rectangle contained in $R_i \cup R_{i+1}$. Let $\sigma = (i, i+1)$ be a transposition. Define L' to be a filling of $\sigma(\mathcal{L})$ by the following operations.

1. Exchange R_i and R_{i+1} with their fillings.

2. Refill the rectangle $\mathcal{R}$ with the unique filling with no ne–chains (Lemma 3.3), preserving the row and column sums of $\mathcal{R}$.

We claim that the filling $\mathcal{L}'$ has no ne–chains. It is clear that any entry outside $\mathcal{R}$ does not change, and $\mathcal{R}$ contains no ne–chains. Let α be a non-zero entry in column c_l outside $\mathcal{R}$. Assume α forms a ne–chain with entry $\beta \neq 0$ in $\mathcal{R}$.

- If $c_l \cap \mathcal{R} \neq \emptyset$, then in L there exists a nonzero entry in the same column as β's. This entry forms a ne-chain with α in L.

- If $c_l \cap \mathcal{R} = \emptyset$, then β is in the longer row of R_i, R_{i+1}. Hence in the filling L there exists a nonzero entry in the longer row of $\mathcal{R}$, which forms a ne-chain with α.

In either case we have a ne-chain in L, which is a contradiction. ☐

Corollary 3.6. *Given a layer polyomino $\mathcal{L}$ with n rows and m columns, vectors $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, $\mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})$ is nonempty if and only if conditions* (3) *and* (4) *hold. Further, if $\vec{r}$ and $\vec{c}$ satisfy* (3) *and* (4) *then the filling of $\mathcal{L}$ with no* ne*–chains (*se*–chains) is unique.*

Proof. Given $\mathcal{L}$, rearrange the rows of $\mathcal{L}$ from large to small to get a polyomino $\mathcal{L}_1$. Then $\mathcal{L}_1$ can be viewed as a layer polyomino rotated 90°. Apply column permutations to transform $\mathcal{L}_1$ to a Ferrers diagram $\mathcal{L}_2$. By Theorem 3.5,

$$\text{av}(\text{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \text{av}(\text{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}_2, \vec{r'}, \vec{c'})), \tag{6}$$

where $\vec{r'}$ and $\vec{c'}$ are obtained from $\vec{r}, \vec{c}$ in the same way when one permutes the rows and columns. From Theorem 3.4 the formula in (6) is non-zero if and only if conditions (3) and (4) hold, in which case the filling is unique. ☐

4 Strong Chains in Layer Polyominoes

In this section we study strong chains as defined in Definition 2.3. We begin by introducing a framework for a bijection on fillings of layer polyominoes. We'll use this framework to prove three distinct results. The first is the equality of numbers of 01-fillings with no strong northeast chains and those with no strong southeast chains. The second shows the symmetry of $(\text{ne}^{\square}, \text{se}^{\square})$ when the column sum is restricted to $\{0, 1\}$. The final result extends the first to $\mathbb{N}$-fillings, with the additional condition that both row and column sums are fixed.

For a polyomino $\mathcal{P}$, let $\mathcal{F}(\mathcal{P})$ be either $\mathbb{N}$- or 01-fillings of $\mathcal{P}$. Let f be an invertible operation so that for any $n \times m$ rectangle $\mathcal{R}$, f induces a bijection $f : \mathcal{F}(\mathcal{R}) \to \mathcal{F}(\mathcal{R})$. Let $\mathcal{L}$ be a layer polyomino with n rows, define $\rho_f : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ recursively as follows. If $\mathcal{L}$ is a rectangle then $\rho_f = f$, otherwise for $L \in \mathcal{F}(\mathcal{L})$,

1. Let $\mathcal{R}_1, \ldots, \mathcal{R}_k$ be the maximal blocks of the consecutive shortest rows, $\mathcal{B}_0, \ldots, \mathcal{B}_k$ be the layer polyominoes between each $\mathcal{R}_i$, and $\mathcal{L}_1$ be the maximal rectangle in $\mathcal{L}$ containing each $\mathcal{R}_i$. See Figure 8 for an illustration of these sets. Let L_1 be the filling of L restricted to $\mathcal{L}_1$.

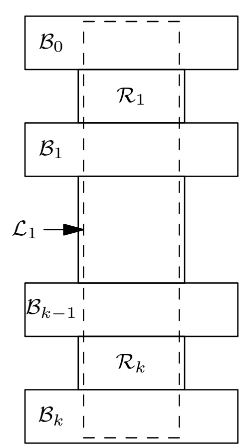


Figure 8: An example of the sets from step 1 of ρ_f.

2. Apply f to L_1.

3. For each i, apply f^{-1} to the current filling in $\mathcal{B}_i \cap \mathcal{L}_1$.

4. For each i, apply ρ_f to the current filling in $\mathcal{B}_i$.

The resulting filling of $\mathcal{L}$ is $\rho_f(L)$.

Proposition 4.1. *The map ρ_f is a well-defined bijection. Additionally, if f preserves row (column) sums, so does ρ_f.*

Proof. By construction ρ_f does not modify the shape of $\mathcal{L}$, thus ρ_f is well defined. The map ρ_f can be inverted by performing each step in reverse, thus ρ_f is a bijection.

If f preserves row (column) sums, then in each step of ρ_f, the row (column) sums are preserved. Therefore, ρ_f preserves row (column) sums. □

Example 4.2. For an $n \times m$ rectangle $\mathcal{R}$, define the map $f_1 : \mathcal{F}_{01}(\mathcal{R}) \to \mathcal{F}_{01}(\mathcal{R})$ so that for $R \in \mathcal{F}_{01}(\mathcal{R})$, $f_1(R)$ leaves empty columns unchanged and reverses each row of R in the nonempty columns. Figure 9 gives an example of f_1. Clearly f_1 preserves row sums.

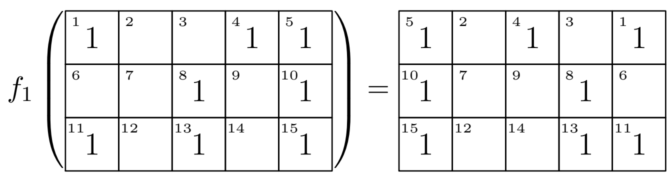


Figure 9: An example of the map f_1.

Proposition 4.3. *The map ρ_{f_1} is a bijection from the set $\mathcal{F}_{01}(\mathcal{L}, \vec{r})$ to itself satisfying $(\rho_{f_1})^{-1} = \rho_{f_1^{-1}} = \rho_{f_1}$.*

Proof. By definition $f_1 = f_1^{-1}$ on 01-fillings of any rectangle. Let $\mathcal{L}$ be a layer polyomino with n rows, $\vec{r} \in \mathbb{N}^n$, and $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r})$, we need to show $\rho_{f_1}(\rho_{f_1}(L)) = L$. It is clearly true when $\mathcal{L}$ is a rectangle.

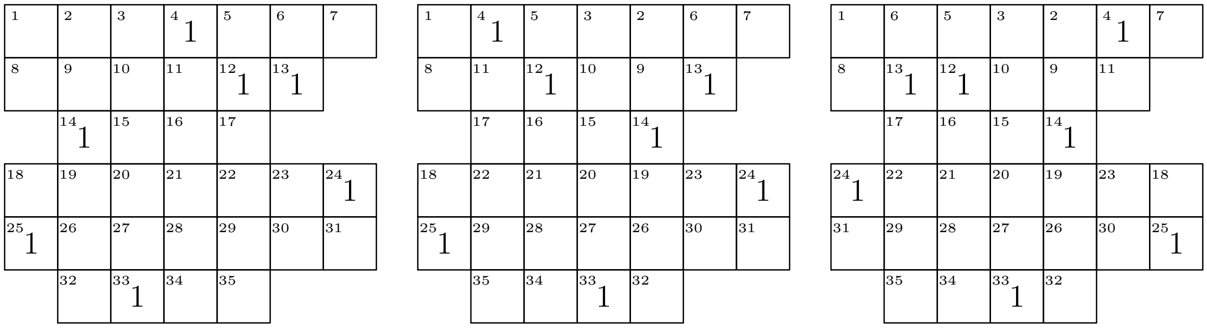
Let $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$ be as in the definition of ρ_{f_1}. Ignoring empty columns of $\mathcal{L}_1$, the map ρ_{f_1} reverses fillings in each $\mathcal{R}_i$, so applying ρ_{f_1} twice leaves fillings in each $\mathcal{R}_i$ unaffected.

Each $\mathcal{B}_j$ is a layer polyomino. Applying f_1 to $\mathcal{L}_1$ and then to $\mathcal{B}_j \cap \mathcal{L}_1$ only changes the location of the empty columns of L in $\mathcal{B}_j$, but does not affect the fillings in the nonempty columns of $\mathcal{B}_j$. Applying ρ_{f_1} on $\mathcal{B}_j$ will not touch the empty columns. Now when we apply the ρ_{f_1} to the whole polyomino $\mathcal{L}$ again, steps 2 and 3 will move the empty columns in $\mathcal{B}_j$ back to their original places, while step 4 will map the current filling on $\mathcal{B}_j$ back to L on $\mathcal{B}_j$, by the inductive hypothesis. □

For a 01-filling L of a layer polyomino $\mathcal{L}$, Figure 10 shows one iteration of the map ρ_{f_1} and the final result. The cells in the polyomino are labeled so one can observe where each cell ends up.

Theorem 4.4. *For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$,*

$$\operatorname{av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \operatorname{av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$



(a) The filling of L (b) First iteration of ρ_{f_1} (c) Result of $\rho_{f_1}(L)$

Figure 10: A demonstration of the map ρ_{f_1}.

Proof. Let f_1 be as defined in Example 4.2. For an $n \times m$ rectangle $\mathcal{R}$ and $\vec{r} \in \mathbb{N}^n$ the restriction $f_1 : \mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{R}, \vec{r})) \to \mathrm{Av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{R}, \vec{r}))$ is clearly well defined. For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$, we show that the restriction $\rho_{f_1} : \mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) \to \mathrm{Av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$ is also well defined. That is, if L has no $\mathrm{ne}^\square$-chains, then ρ_{f_1} has no $\mathrm{se}^\square$-chains.

We proceed by induction on the number of rows of $\mathcal{L}$. If $\mathcal{L}$ has only one row or is a rectangle, then the claim is true. In general, let $L \in \mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$ and $\mathcal{R}_i$, $\mathcal{B}_j$, $\mathcal{L}_1$ and L_1 be as in the definition of ρ_{f_1}. Within L_1, there are no $\mathrm{ne}^\square$-chains so that $f_1(L_1)$ has no $\mathrm{se}^\square$-chains. For any nonempty cell α in some $\mathcal{R}_i$ the cells to the upper left and lower right of α are empty in $f_1(L_1)$. Since f_1 fixes empty columns, the empty cells will remain empty in the final filling $\rho_{f_1}(L)$. Thus α forms no $\mathrm{se}^\square$-chains in $\rho_{f_1}(L)$.

For cells in $\mathcal{B}_j$ for some j, as observed before, applying f_1 to L_1 and then to $L_1 \cap \mathcal{B}_j$ will not change the filling $L \cap \mathcal{B}_j$ in the nonempty columns of $\mathcal{B}_j$. Hence there are no $\mathrm{ne}^\square$-chains after steps 2 and 3. By induction, applying ρ_{f_1} to $\mathcal{B}_j$ yields a filling with no $\mathrm{se}^\square$-chains.

Therefore, $\rho_{f_1}(\mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))) \subseteq \mathrm{Av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$, and hence

$$\mathrm{av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) \leq \mathrm{av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$

The reverse direction is proved similarly. In conclusion, ρ_{f_1} is a bijection from $\mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r})$ to $\mathrm{Av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$. □

If one fixes both row sum $\vec{r}$ and column sum $\vec{c}$, then $\mathrm{av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c}))$ may not equal $\mathrm{av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c}))$, as shown in the polyomino in Figure 11 with $\vec{r} = (1, 1, 2)$ and $\vec{c} = (1, 2, 1)$. It is easy to check that $\mathrm{av}(\mathrm{ne}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) = 0$ and $\mathrm{av}(\mathrm{se}^\square, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) = 1$.

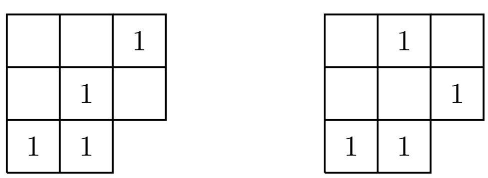


Figure 11: Fillings with $\vec{r} = (1, 1, 2)$ and $\vec{c} = (1, 2, 1)$.

The following theorem was first proved by Phillipson, Yan and Yeh [9] by induction on the generating functions. Now we give a bijective proof by using the map ρ_{f_1}. In the statement of Theorem 4.5 statistics $\mathrm{ne}^\square$ and $\mathrm{se}^\square$ represent the numbers of strong ne- and se-chains.

Theorem 4.5. *For a layer polyomino $\mathcal{L}$ with n rows and m columns, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \{0,1\}^m$, the map ρ_{f_1} restricted to $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ is a bijection that maps $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ to itself and carries the statistics $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ to $(\mathrm{se}^{\square}, \mathrm{ne}^{\square})$, where f_1 is defined as in Example 4.2. Consequently, the distribution of the joint statistic $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ is symmetric in $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$.*

Proof. For an $n \times m$ rectangle $\mathcal{R}$, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \{0,1\}^m$, the restriction of f_1 to $\mathcal{F}_{01}(\mathcal{R}, \vec{r}, \vec{c})$ is well defined as f_1 fixes empty columns, and hence preserves the column sums when $\vec{c} \in \{0,1\}^m$. Also, f_1 exchanges $\mathrm{ne}^{\square}$ and $\mathrm{se}^{\square}$-chains in $\mathcal{R}$.

We will show that ρ_{f_1} exchanges the numbers of $\mathrm{ne}^{\square}$ and $\mathrm{se}^{\square}$-chains for fillings in $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$. Again we proceed by induction. The claim is obvious if $\mathcal{L}$ has only one row or is a rectangle. Assume it is true for all layer polyominoes with less than n rows. For $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ set $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$, L_1 to be as in the definition of ρ_{f_1}.

First we show that each $\mathrm{ne}^{\square}$-chain (resp, $\mathrm{se}^{\square}$–chain) of L_1 not completely contained in some $\mathcal{B}_i$ has a corresponding $\mathrm{se}^{\square}$-chain (resp, $\mathrm{ne}^{\square}$–chain) in $\rho_{f_1}(L)$. Explicitly, let entries at cells α, β be such a $\mathrm{ne}^{\square}$-chain, where the columns of α, β are the k_1^{th} and k_2^{th} nonempty columns of L_1, counting from left, then Step 2 of ρ_{f_1} maps them to a $\mathrm{se}^{\square}$-chain with 1-cells γ and δ, in the k_1^{th} and k_2^{th} nonempty columns of $f_1(L_1)$, counting from right. We have three cases.

1. Both α and β are contained in (possibly different) $\mathcal{R}_i$'s.
2. One of α, β is contained in $\mathcal{R}_i$, and the other is in $\mathcal{B}_j$.
3. The cell α is in $\mathcal{B}_i$ and β in $\mathcal{B}_j$, with $i \neq j$.

For Case 1, γ and δ are in the same $\mathcal{R}_i$ as α and β, respectively, and are not changed further by steps 3,4 of ρ_{f_1}. Thus (γ, δ) remains a $\mathrm{se}^{\square}$-chain.

For Case 2, without loss of generality, assume $\alpha \in \mathcal{R}_i$ and $\beta \in \mathcal{B}_j$. Then $\gamma \in \mathcal{R}_i$ and will not be changed further. The cell δ may be changed in Steps 3 and 4 of ρ_{f_1}. However, the operations on both steps 3 and 4 preserve the column sum, hence in the final filling there is a unique 1-cell in $\mathcal{B}_j$ that lies in the same column as δ. It forms a $\mathrm{se}^{\square}$-chain with γ.

For Case 3, we have $\gamma \in \mathcal{B}_i$ and $\delta \in \mathcal{B}_j$. By the same argument as in Case 2, after steps 3 and 4 there is a unique 1-cell in $\mathcal{B}_i$ that lies in the same column as γ, and a unique 1-cell in $\mathcal{B}_j$ that lies in the same column as δ. These two 1-cells form a $\mathrm{se}^{\square}$-chain.

Applying the same argument in reverse, we conclude that the $\mathrm{ne}^{\square}$-chains ($\mathrm{se}^{\square}$-chains) of L that are not completely in some $\mathcal{B}_i$ are in one-to-one correspondence with the $\mathrm{se}^{\square}$-chains ($\mathrm{ne}^{\square}$-chains) of $\rho_{f_1}(L)$ not completely in some $\mathcal{B}_i$.

Finally we look at the strong chains inside $\mathcal{B}_i$ for some i. As observed before, steps 2 and 3 of ρ_{f_1} do not change the filling on the nonempty columns of $\mathcal{B}_i$. Hence after these two steps, the number of strong chains in each $\mathcal{B}_i$ doesn't change. When one applies step 4, by inductive hypothesis, ρ_{f_1} on $\mathcal{B}_i$ is a bijection that maps the statistics $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ of fillings of $\mathcal{B}_i$ to the statistics $(\mathrm{se}^{\square}, \mathrm{ne}^{\square})$ on $\mathcal{B}_i$.

Theorem 4.5 follows from combining all the above cases.

□

The final result is an analog of Theorem 4.4 on $\mathbb{N}$-fillings. We show that over $\mathbb{N}$–fillings of layer polyominoes with fixed row and column sums, the number of fillings with no $\mathrm{ne}^{\square}$–chains is the same as that with no $\mathrm{se}^{\square}$–chains.

Theorem 4.6. *For a layer polyomino $\mathcal{L}$ with n rows and m columns, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \mathbb{N}^m$,*

$$\mathrm{av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{se}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})).$$

Proof. For an $n \times m$ rectangle $\mathcal{R}$, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, define the map

$$f_2 : \mathrm{Av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(R, \vec{r}, \vec{c})) \to \mathrm{Av}(\mathrm{se}, \mathcal{F}_{\mathbb{N}}(R, \vec{r}, \vec{c}))$$

that maps the unique filling of $\mathcal{R}$ with no ne-chains to the unique filling with no se-chains, as in Lemma 3.3. For a layer polyomino $\mathcal{L}$ with row sums $\vec{r}$ and column sums $\vec{c}$, we show the restriction $\rho_{f_2} : \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) \to \mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ is also well defined.

The proof is again by induction and analogous to that of Theorem 4.4. Let $L \in \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ and $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$, L_1 be as in the definition of ρ_{f_2}. Step 2 of ρ_{f_2} maps the filling L_1 to one with no $\mathrm{se}^{\square}$-chains. For any nonempty cell α in some $\mathcal{R}_i$, the columns to the upper right and lower left are empty. Since f_2 preserves column sums, these areas will remain empty in $\rho_{f_2}(L)$.

For each $\mathcal{B}_i$, we claim that after steps 2 and 3 of ρ_{f_2}, the filling on $\mathcal{B}_i$ contains no $\mathrm{ne}^{\square}$-chains. To see this, note that by definition of f_2, there is no $\mathrm{ne}^{\square}$-chain inside $\mathcal{L}_1 \cap \mathcal{B}_i$. Clearly there is no $\mathrm{ne}^{\square}$-chains containing two cells in $\mathcal{B}_i \setminus \mathcal{L}_1$. If there exists a $\mathrm{ne}^{\square}$-chain on $\mathcal{B}_i$ containing cells α, β with $\alpha \notin \mathcal{L}_1$ and $\beta \in \mathcal{L}_1$, since both operations in steps 2 and 3 preserve the row sums of $\mathcal{L}_1 \cap \mathcal{B}_i$, there must be a nonempty cell in L that lies in the same row as β in $\mathcal{L}_1 \cap \mathcal{B}_i$. Such a cell and α form a $\mathrm{ne}^{\square}$-chain in L, a contradiction.

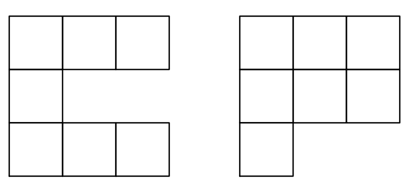
Thus ρ_{f_2} is well defined and is an injection from $\mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ to $\mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$. The reverse inclusion is similarly proved, which implies Theorem 4.6. □

REMARKS

1. Unlike 01-fillings, for general $\mathcal{L}$, $\mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ contains more than one element.

2. In general it's not true that for a given f, $(\rho_f)^{-1} = \rho_{f^{-1}}$. However, this is the case for the f_1 and f_2 we used. It's unclear what conditions on f would guarantee this, but it's an interesting occurrence.

3. It is natural to ask if we can flip adjacent rows while preserving the number of $\mathrm{ne}^{\square}$–chains in layer polyominoes. The answer is no. Consider the polyominoes in Figure 12 with row and column sums $(1, 1, 1)$. The left polyomino has 2 fillings with no $\mathrm{ne}^{\square}$–chains whereas the right



(a) (b)

Figure 12: Layer polyominoes with one row flipped.

has only 1. In fact, this example shows that the joint distribution of $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ is dependent on the order of the rows in layer polyominoes.

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